Introduction to Finite Frame Theory

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1 Why Frames?

The Fourier transform has been a major tool in analysis for over 100 years. However, it solely provides frequency information, and hides (in its phases) information concerning the moment of emission and duration of a signal. D. Gabor resolved this problem in 1946 [93] by introducing a fundamental new approach to signal decomposition. Gabor's approach quickly became the paradigm for this area, because it provided resilience to additive noise, resilience to quantization, resilience to transmission losses as well as an ability to capture important signal characteristics. Unbeknownst to Gabor, he had discovered the fundamental properties of a frame without any of the formalism. In 1952, Duffin and Schaeffer [80] were working on some deep problems in non-harmonic Fourier series for which they required a formal structure for working with highly over-complete families of exponential functions in $L^2[0,1]$. For this, they introduced the notion of a *Hilbert space frame*, for which Gabor's approach is now a special case, falling into the area of *time-frequency analysis* [98]. Much later – in the late 1980's – the fundamental concept of frames was revived by Daubechies, Grossman and Mayer [77] (see also [76]), who showed its importance for data processing.

Traditionally, frames were used in signal and image processing, non-harmonic Fourier series, data compression, and sampling theory. But today, frame theory has ever increasing applications to problems in both pure and applied mathematics,

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physics, engineering, computer science, to name a few. Several of these applications will be investigated in this book. Since applications mainly require frames in finite-dimensional spaces, this will be our focus. In this situation, a frame is a spanning set of vectors – which are generally *redundant* (*over-complete*) requiring control of its condition numbers. Thus a typical frame possesses more frame vectors than the dimension of the space, and each vector in the space will have infinitely many representations with respect to the frame. But it will also have one natural representation given by a special class of scalars called the *frame coefficients* of the vector. It is this *redundancy of frames* which is key to their significance for applications.

The role of redundancy varies depending on the requirements of the applications at hand. First, redundancy gives greater design *flexibility* which allows frames to be constructed to fit a particular problem in a manner not possible by a set of linearly independent vectors. For instance, in areas such as quantum tomography, classes of orthonormal bases with the property that the modulus of the inner products of vectors from different bases are a constant are required. A second example comes from speech recognition, when a vector needs to be determined by the absolute value of the frame coefficients (up to a phase factor). A second major advantage of redundancy is *robustness*. By spreading the information over a wider range of vectors, resilience against losses (*erasures*) can be achieved, which are, for instance, a severe problem in wireless sensor networks for transmission losses or when sensors are intermittently fading out, modeling the brain where memory cells are dying out. A further advantage of spreading information over a wider range of vectors is to mitigate the effects of noise in the signal.

This represents a tiny fraction of the theory and applications of frame theory that you will encounter in this book. New theoretical insights and novel applications are continually arising due to the fact that the underlying principles of frame theory are basic ideas which are fundamental to a wide canon of areas of research. In this sense, frame theory might be regarded as partly belonging to applied harmonic analysis, functional analysis, operator theory as well as numerical linear algebra and matrix theory.

1.1 The Role of Decompositions and Expansions

Focussing on the finite-dimensional situation, let x be given data which we assume to belong to some real or complex N-dimensional Hilbert space $\mathscr{H}^N$. Further, let $(\varphi_i)_{i=1}^M$ be a representation system (i.e. a spanning set) in $\mathscr{H}^N$, which might be chosen from an existing catalogue, designed depending on the type of data we are facing, or learned from sample sets of the data.

One common approach to data processing consists in the *decomposition* of the data x according to the system $(\varphi_i)_{i=1}^M$ by considering the map

$$x \mapsto (\langle x, \varphi_i \rangle)_{i=1}^M.$$

As we will see, the generated sequence $(\langle x, \varphi_i \rangle)_{i=1}^M$ belonging to $\ell_2(\{1,\dots,M\})$ can then be used, for instance, for transmission of x. Also, careful choice of the representation system enables us to solve a variety of analysis tasks. As an example, under certain conditions the positions and orientations of edges of an image x are determined by those indices $i \in \{1,\dots,M\}$ belonging to the largest coefficients in magnitude $|\langle x, \varphi_i \rangle|$, i.e., by hard thresholding, in the case that $(\varphi_i)_{i=1}^M$ is a shearlet system (see [116]). Finally, the sequence $(\langle x, \varphi_i \rangle)_{i=1}^M$ allows compression of x, which is in fact the heart of the new JPEG2000 compression standard when choosing $(\varphi_i)_{i=1}^M$ to be a wavelet system [141].

An accompanying approach is the *expansion* of the data x by considering sequences $(c_i)_{i=1}^M$ satisfying

$$x = \sum_{i=1}^M c_i \varphi_i.$$

It is well known that suitably chosen representation systems allow sparse sequences $(c_i)_{i=1}^M$ in the sense that $\|c\|_0 = \#\{i : c_i \neq 0\}$ is small. For example, certain wavelet systems typically sparsify natural images in this sense (see for example [78, 123, 134] and the references therein). This observation is key to allowing the application of the abundance of existing sparsity methodologies such as Compressed Sensing [87] to x. In contrast to this viewpoint which assumes x as explicitly given, the approach of expanding the data is also highly beneficial in the case where x is only implicitly given, which is, for instance, the problem all PDE solvers face. Hence, using $(\varphi_i)_{i=1}^M$ as a generating system for the trial space, the PDE solvers task reduces to computing $(c_i)_{i=1}^M$ which is advantageous for deriving efficient solvers provided that – as before – a sparse sequence does exist (see, e.g., [107, 74]).

1.2 Beyond Orthonormal Bases

To choose the representation system $(\varphi_i)_{i=1}^N$ to form an orthonormal basis for $\mathscr{H}^N$ is the standard choice. However, the linear independence of such a system causes a variety of problems for the aforementioned applications.

Starting with the *decomposition* viewpoint, using $(\langle x, \varphi_i \rangle)_{i=1}^N$ for transmission is far from being robust to erasures, since the erasure of only a single coefficient causes a true information loss. Also, for analysis tasks orthonormal bases are far from being advantageous, since they do not allow any flexibility in design, which is for instance needed for the design of directional representation systems. In fact, it is conceivable that no orthonormal basis with paralleling properties such as curvelets or shearlets does exist. A task benefitting from linear independence is compression, which naturally requires a minimal number of coefficients.

Also, from an *expansion* point of view, the utilization of orthonormal bases is not advisable. A particular problem affecting sparsity methodologies as well as the utilization for PDE solvers is the uniqueness of the sequence $(c_i)_{i=1}^M$. This nonflexibility prohibits the search for a sparse coefficient sequence.

It is evident that those problems can be tackled by allowing the system $(\varphi_i)_{i=1}^M$ to be redundant. Certainly, numerical stability issues in the typical processing of data

$$x \mapsto (\langle x, \varphi_i \rangle)_{i=1}^M \mapsto \sum_{i=1}^M \langle x, \varphi_i \rangle \tilde{\varphi}_i \approx x$$

with an adapted system $(\tilde{\varphi}_i)_{i=1}^M$ have to be taken into account. This leads naturally to the notion of a (*Hilbert space*) *frame*. The main idea is to have a controlled norm equivalence between the data x and the sequence of coefficients $(\langle x, \varphi_i \rangle)_{i=1}^M$.

The area of frame theory has very close relations to other research fields in both pure and applied mathematics. General (Hilbert space) frame theory – in particular, including the infinite-dimensional situation – intersects functional analysis and operator theory. It also bears close relations to the novel area of applied harmonic analysis, in which the design of representation systems – typically by a careful partitioning of the Fourier domain – is one major objective. Some researchers even consider frame theory as belonging to this area. Restricting to the finite-dimensional situation – in which customarily the term *finite frame theory* is used – the classical areas of matrix theory and numerical linear algebra have close intersections, but also, for instance, the novel area of Compressed Sensing as already pointed out.

Nowadays, frames have established themselves as a standard notion in applied mathematics, computer science, and engineering. Finite frame theory deserves special attention due to its importance for applications, and might be even considered a research area of its own. This is also the reason why this book specifically focusses on finite frame theory. The subsequent chapters will show the diversity of this rich and vivid research area to date ranging from the development of frameworks to analyze specific properties of frames, the design of different classes of frames to various applications of frames and also extensions of the notion of a frame.

1.3 Outline

In the sequel, in Section 2 we first provide some background information on Hilbert space theory and operator theory to make this book self-contained. Frames are then subsequently introduced in Section 3, followed by a discussion of the four main operators associated with a frame, namely the analysis, synthesis, frame, and Grammian operator (see Section 4). Reconstruction results and algorithms naturally including the notion of a dual frame is the focus of Section 5. This is followed by the presentation of different constructions of tight as well as non-tight frames (Section 6), and a discussion of some crucial properties of frames, in particular, their spanning properties, the redundancy of a frame, and equivalence relations among frames in Section 7. This chapter is concluded by brief introductions to diverse applications and extensions of frames (Sections 8 and 9).

2 Background Material

Let us start by recalling some basic definitions and results from Hilbert space theory and operator theory, which will be required for all subsequent chapters. We do not include the proofs of the presented results, and refer to the standard literature such as, for instance, [153] for Hilbert space theory and [71, 105, 130] for operator theory. We would like to emphasize that all following results are solely stated in the finite dimensional setting, which is the focus of this book.

2.1 Review of Basics from Hilbert Space Theory

Letting N be a positive integer, we denote by $\mathscr{H}^N$ a real or complex N-dimensional Hilbert space. This will be the space considered throughout this book. Sometimes, if it is convenient, we identify $\mathscr{H}^N$ with $\mathbb{R}^N$ or $\mathbb{C}^N$. By $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ we denote the inner product on $\mathscr{H}^N$ and its corresponding norm, respectively.

Let us now start with the origin of frame theory, which is the notion of an orthonormal basis. Alongside, we recall the basic definitions we will also require in the sequel.

Definition 1. A vector $x \in \mathscr{H}^N$ is called *normalized* if $\|x\| = 1$. Two vectors $x,y \in \mathscr{H}^N$ are called *orthogonal* if $\langle x,y\rangle = 0$. A system $(e_i)_{i=1}^k$ of vectors in $\mathscr{H}^N$ is called

(a) *complete* (or a *spanning set*) if $\operatorname{span}\{e_i\}_{i=1}^k = \mathscr{H}^N$.
(b) *orthogonal* if for all $i \neq j$, the vectors e_i and e_j are orthogonal.
(c) *orthonormal* if it is orthogonal and each e_i is normalized.
(e) *orthonormal basis* for $\mathscr{H}^N$ if it is complete and orthonormal.

A fundamental result in Hilbert space theory is *Parseval's Identity*.

Proposition 1 (Parseval's Identity). *If $(e_i)_{i=1}^N$ is an orthonormal basis for $\mathscr{H}^N$, then, for every $x \in \mathscr{H}^N$, we have*

$$\|x\|^2 = \sum_{i=1}^N |\langle x,e_i\rangle|^2.$$

Interpreting this identity from a signal processing point of view, it implies that the energy of the signal is preserved under the map $x \mapsto (\langle x,e_i\rangle)_{i=1}^N$ which we will later refer to as the analysis map. We would also like to mention at this point, that this identity is not only satisfied by orthonormal bases. In fact, redundant systems ("non-bases") such as $(e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \dots, \frac{1}{\sqrt{N}}e_N, \dots, \frac{1}{\sqrt{N}}e_N)$ also satisfy this inequality, and will later be coined *Parseval frames.*

Parseval's identity has the following implication, which shows that a vector x can be recovered from the coefficients $(\langle x,e_i\rangle)_{i=1}^N$ by a simple procedure. Thus, from an application point of view, this result can also be interpreted as a reconstruction formula.

Corollary 1. *If $(e_i)_{i=1}^N$ is an orthonormal basis for $\mathscr{H}^N$, then, for every $x \in \mathscr{H}^N$, we have*

$$x = \sum_{i=1}^{N} \langle x, e_i \rangle e_i.$$

Next, we present a series of identities and inequalities, which are basics exploited in various proofs.

Proposition 2. *Let $x, \tilde{x} \in \mathscr{H}^N$.*

(i) Cauchy-Schwartz Inequality. *We have*

$$|\langle x, \tilde{x} \rangle| \leq \|x\| \|\tilde{x}\|,$$

with equality if and only if $x = c\tilde{x}$ for some constant c.

(ii) Triangle Inequality. *We have*

$$\|x + \tilde{x}\| \leq \|x\| + \|\tilde{x}\|.$$

(iii) Polarization Identity (Real Form). *If $\mathscr{H}^N$ is real, then*

$$\langle x, \tilde{x} \rangle = \frac{1}{4} \left[\|x + \tilde{x}\|^2 - \|x - \tilde{x}\|^2 \right].$$

(iv) Polarization Identity (Complex Form). *If $\mathscr{H}^N$ is complex, then*

$$\langle x, \tilde{x} \rangle = \frac{1}{4} \left[\|x + \tilde{x}\|^2 - \|x - \tilde{x}\|^2 \right] + \frac{i}{4} \left[\|x + i\tilde{x}\|^2 - \|x - i\tilde{x}\|^2 \right].$$

(v) Pythagorean Theorem. *Given pairwise orthogonal vectors $(x_i)_{i=1}^M \in \mathscr{H}^N$, we have*

$$\left\| \sum_{i=1}^{M} x_i \right\|^2 = \sum_{i=1}^{M} \|x_i\|^2.$$

We next turn to considering subspaces in $\mathscr{H}^N$, again starting with the basic notations and definitions.

Definition 2. Let $\mathscr{W}, \mathscr{V}$ be subspaces of $\mathscr{H}^N$.

(a) A vector $x \in \mathscr{H}^N$ is called *orthogonal to* $\mathscr{W}$ (denoted by $x \perp \mathscr{W}$), if

$$\langle x, \tilde{x} \rangle = 0 \quad \text{for all } \tilde{x} \in \mathscr{W}.$$

The *orthogonal complement* of $\mathscr{W}$ is then defined by

$$\mathscr{W}^\perp = \{x \in \mathscr{H}^N : x \perp \mathscr{W}\}.$$

(b) The subspaces $\mathscr{W}$ and $\mathscr{V}$ are called *orthogonal subspaces* (denoted by $\mathscr{W} \perp \mathscr{V}$), if $\mathscr{W} \subset \mathscr{V}^\perp$ (or, equivalently, $\mathscr{V} \subset \mathscr{W}^\perp$).

The notion of *orthogonal direct sums*, which will play an essential role in Chapter [166], can be regarded as a generalization of Parseval's identity (Proposition 1).

Definition 3. Let $(\mathscr{W}_i)_{i=1}^M$ be a family of subspaces of $\mathscr{H}^N$. Then their *orthogonal direct sum* is defined as the space

$$\left(\sum_{i=1}^M \oplus \mathscr{W}_i\right)_{\ell^2} := \mathscr{W}_1 \times \ldots \times \mathscr{W}_M$$

with inner product defined by

$$\langle x, \tilde{x} \rangle = \sum_{i=1}^M \langle x_i, \tilde{x}_i \rangle \quad \text{for all } x = (x_i)_{i=1}^M,\, \tilde{x} = (\tilde{x}_i)_{i=1}^M \in \left(\sum_{i=1}^M \oplus \mathscr{W}_i\right)_{\ell^2}.$$

The extension of Parseval's identity can be seen when choosing $\tilde{x} = x$ yielding $\|x\|^2 = \sum_{i=1}^M \|x_i\|^2$.

2.2 Review of Basics from Operator Theory

We next introduce the basic results from operator theory used throughout this book. We first recall that each operator has an associated matrix representation.

Definition 4. Let $T : \mathscr{H}^N \to \mathscr{H}^K$ be a linear operator, let $(e_i)_{i=1}^N$ be an orthonormal basis for $\mathscr{H}^N$, and let $(g_i)_{i=1}^K$ be an orthonormal basis for $\mathscr{H}^K$. Then the *matrix representation of* T (*with respect to the orthonormal bases* $(e_i)_{i=1}^N$ *and* $(g_i)_{i=1}^K$) is a matrix of size $K \times N$ and is given by $A = (a_{ij})_{i=1,\, j=1}^{K\ ,\ N}$, where

$$a_{ij} = \langle Te_j, g_i \rangle.$$

For all $x \in \mathscr{H}^N$ with $c = (\langle x, e_i \rangle)_{i=1}^N$ we have

$$Tx = Ac.$$

2.2.1 Invertibility

We start with the following definition.

Definition 5. Let $T : \mathscr{H}^N \to \mathscr{H}^K$ be a linear operator.

(a) The *kernel* of T is defined by $\ker T := \{x \in \mathscr{H}^N : Tx = 0\}$. Its *range* is $\operatorname{ran} T := \{Tx : x \in \mathscr{H}^N\}$, sometimes also called *image* and denoted by $\operatorname{im} T$. The *rank of* T, $\operatorname{rank} T$, is the dimension of the range of T.

(b) The operator T is called *injective* (or *one-to-one*), if $\ker T = \{0\}$, and *surjective* (or *onto*), if $\operatorname{ran} T = \mathscr{H}^K$. It is called *bijective (or invertible)*, if T is both injective and surjective.
(c) The *adjoint operator* $T^* : \mathscr{H}^K \to \mathscr{H}^N$ is defined by

$$\langle Tx, \tilde{x}\rangle = \langle x, T^*\tilde{x}\rangle \quad \text{for all } x \in \mathscr{H}^N \text{ and } \tilde{x} \in \mathscr{H}^K.$$

(d) The *norm* of T is defined by

$$\|T\| := \sup\{\|Tx\| : \|x\| = 1\}.$$

The next result states several relations between these notions.

Proposition 3. *(i) Let $T : \mathscr{H}^N \to \mathscr{H}^K$ be a linear operator. Then*

$$\dim \mathscr{H}^N = N = \dim \ker T + \operatorname{rank} T.$$

Moreover, if T is injective, then T^*T is also injective.
(ii) Let $T : \mathscr{H}^N \to \mathscr{H}^N$ be a linear operator. Then T is injective if and only if it is surjective. Moreover , $\ker T = (\operatorname{ran} T^*)^\perp$*, and hence*

$$\mathscr{H}^N = \ker T \oplus \operatorname{ran} T^*.$$

If $T : \mathscr{H}^N \to \mathscr{H}^N$ is an injective operator, then T is obviously invertible. If an operator $T : \mathscr{H}^N \to \mathscr{H}^K$ is not injective, we can make T injective by restricting it to $(\ker T)^\perp$. However, $T|_{(\ker T)^\perp}$ might still not be invertible, since it does not need to be surjective. This can be ensured by considering the operator $T : (\ker T)^\perp \to \operatorname{ran} T$, which is now invertible.

The Moore-Penrose inverse of an injective operator provides a one-sided inverse for the operator.

Definition 6. Let $T : \mathscr{H}^N \to \mathscr{H}^K$ be an injective, linear operator. The *Moore-Penrose inverse* of T, $T^\dagger$, is defined by

$$T^\dagger = (T^*T)^{-1}T^*.$$

It is immediate to prove invertibility from the left as stated in the following result.

Proposition 4. *If $T : \mathscr{H}^N \to \mathscr{H}^K$ is an injective, linear operator, then $T^\dagger T = Id$.*

Thus, $T^\dagger$ plays the role of the inverse on $\operatorname{ran} T$ – not on all of $\mathscr{H}^K$. It projects a vector from $\mathscr{H}^K$ onto $\operatorname{ran} T$ and then inverts the operator on this subspace.

A more general notion of this inverse is called the *pseudo-inverse*, which can be applied to a non-injective operator. It, in fact, adds one more step to the action of $T^\dagger$ by first restricting to $(\ker T)^\perp$ to enforce injectivity of the operator followed by application of the Moore-Penrose inverse of this new operator. This pseudo-inverse can be derived from the singular value decomposition. Recalling that by fixing orthonormal bases of the domain and range of a linear operator we derive an associated unique matrix representation, we begin by stating this decomposition in terms of a matrix.

Theorem 1. *Let A be an $M \times N$ matrix. Then there exist an $M \times M$ unitary matrix U (see Definition 9), an $N \times N$ unitary matrix V, and an $M \times N$ diagonal matrix Σ with nonnegative, decreasing real entries on the diagonal such that*

$$A = U\Sigma V^*.$$

Hereby, an $M \times N$ diagonal matrix with $M \neq N$ is an $M \times N$ matrix $(a_{ij})_{i=1,\,j=1}^{M,\ N}$ with $a_{ij} = 0$ for $i \neq j$.

Definition 7. Let A be an $M \times N$ matrix, and let U, Σ, and V be chosen as in Theorem 1. Then $A = U\Sigma V^*$ is called the *singular value decomposition (SVD)* of A. The column vectors of U are called the *left singular vectors*, and the column vectors of V are referred to as the *right singular vectors* of A.

The pseudo-inverse A^+ of A can be deduced from the SVD in the following way.

Theorem 2. *Let A be an $M \times N$ matrix, and let $A = U\Sigma V^*$ be its singular value decomposition. Then*

$$A^+ = V\Sigma^+ U^*,$$

where Σ^+ is the $N \times M$ diagonal matrix arising from Σ^* by inverting the non-zero (diagonal) entries.

2.2.2 Riesz Bases

In the previous subsection, we already recalled the notion of an orthonormal basis. However, sometimes the requirement of orthonormality is too strong, but uniqueness of a decomposition as well as stability shall be retained. The notion of a Riesz basis, which we next introduce, satisfies these desiderata.

Definition 8. A family of vectors $(\varphi_i)_{i=1}^N$ in a Hilbert space $\mathscr{H}^N$ is a *Riesz basis* with *lower* (respectively, *upper*) *Riesz bounds* A (resp. B), if, for all scalars $(a_i)_{i=1}^N$, we have

$$A\sum_{i=1}^N |a_i|^2 \leq \left\| \sum_{i=1}^N a_i\varphi_i \right\|^2 \leq B\sum_{i=1}^N |a_i|^2.$$

The following result is immediate from the definition.

Proposition 5. *Let $(\varphi_i)_{i=1}^N$ be a family of vectors. Then the following conditions are equivalent.*

(i) $(\varphi_i)_{i=1}^N$ is a Riesz basis for $\mathscr{H}^N$ with Riesz bounds A and B.
(ii) For any orthonormal basis $(e_i)_{i=1}^N$ for $\mathscr{H}^N$, the operator T on $\mathscr{H}^N$ given by $Te_i = \varphi_i$ for all $i = 1, 2, \ldots, N$ is an invertible operator with $\|T\|^2 \leq B$ and $\|T^{-1}\|^{-2} \geq A$.

2.2.3 Diagonalization

Next, we continue our list of important properties of linear operators.

Definition 9. A linear operator $T : \mathscr{H}^N \to \mathscr{H}^K$ is called

(a) *self-adjoint*, if $\mathscr{H}^N = \mathscr{H}^K$ and $T = T^*$.
(b) *normal*, if $T^*T = TT^*$.
(c) *an isometry*, if $\|Tx\| = \|x\|$ for all $x \in \mathscr{H}^N$.
(d) *a co-isometry*, if T^* is an isometry.
(e) *positive*, if $\mathscr{H}^N = \mathscr{H}^K$, T is self-adjoint, and $\langle Tx, x\rangle \geq 0$ for all $x \in \mathscr{H}^N$.
(f) *unitary*, if it is a surjective isometry.

From the variety of basic relations and results of those notions, the next proposition presents a selection of those which will be required in the sequel.

Proposition 6. *Let $T : \mathscr{H}^N \to \mathscr{H}^K$ be a linear operator.*

(i) We have $\|T^*T\| = \|T\|^2$, and T^*T and TT^* are self-adjoint.
(ii) If $N = K$, the following conditions are equivalent.

(1) T is self-adjoint.
(2) $\langle Tx, \tilde{x}\rangle = \langle x, T\tilde{x}\rangle$ for all $x, \tilde{x} \in \mathscr{H}^N$.
(3) If $\mathscr{H}^N$ is complex, $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in \mathscr{H}^N$.

(iii) The following conditions are equivalent.

(1) T is an isometry.
(2) $T^*T = Id$.
(3) $\langle Tx, T\tilde{x}\rangle = \langle x, \tilde{x}\rangle$ for all $x, \tilde{x} \in \mathscr{H}^N$.

(iv) The following conditions are equivalent.

(1) T is unitary.
(2) T and T^* are isometric.
(3) $TT^* = Id$ and $T^*T = Id$.

(v) If U is a unitary operator, then $\|UT\| = \|T\| = \|TU\|$.

Diagonalizations of operators are frequently utilized to derive an understanding of the action of an operator. The following definitions lay the groundwork for this theory.

Definition 10. Let $T : \mathscr{H}^N \to \mathscr{H}^N$ be a linear operator. A non-zero vector $x \in \mathscr{H}^N$ is an *eigenvector* of T with *eigenvalue* λ, if $Tx = \lambda x$. The operator T is called *orthogonally diagonalizable*, if there exists an orthonormal basis $(e_i)_{i=1}^N$ of $\mathscr{H}^N$ consisting of eigenvectors of T.

We start with an easy observation.

Proposition 7. *For any linear operator $T : \mathscr{H}^N \to \mathscr{H}^K$, the non-zero eigenvalues of T^*T and TT^* are the same.*

If the operator is unitary, self-adjoint or positive, we have more information on the eigenvalues stated in the next result, which follows immediately from Proposition 6.

Corollary 2. *Let $T : \mathscr{H}^N \to \mathscr{H}^N$ be a linear operator.*

(i) If T is unitary, then its eigenvalues have modulus one.
(ii) If T is self-adjoint, then its eigenvalues are real.
(iii) If T is positive, then its eigenvalues are non-negative.

This fact allows us to introduce a condition number associated with each invertible positive operator.

Definition 11. Let $T : \mathscr{H}^N \to \mathscr{H}^N$ be an invertible positive operator with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. Then its *condition number* is defined by $\frac{\lambda_1}{\lambda_N}$.

We next state a fundamental result in operator theory which has its analogue in the infinite-dimensional setting called the *spectral theorem*.

Theorem 3. *Let $\mathscr{H}^N$ be complex and let $T : \mathscr{H}^N \to \mathscr{H}^N$ be a linear operator. Then the following conditions are equivalent.*

(i) T is normal.
(ii) T is orthogonally diagonizable.
(iii) There exists a diagonal matrix representation of T.
(iv) There exist an orthonormal basis $(e_i)_{i=1}^N$ of $\mathscr{H}^N$ and values $\lambda_1, \ldots, \lambda_N$ such that

$$Tx = \sum_{i=1}^N \lambda_i \langle x, e_i \rangle e_i \quad \textit{for all } x \in \mathscr{H}^N.$$

Moreover,

$$\|T\|^2 = \max_{1 \leq i \leq N} \lambda_i.$$

Since every self-adjoint operator is normal we obtain the following corollary (which is independent of whether $\mathscr{H}^N$ is real or complex).

Corollary 3. *A self-adjoint operator is orthogonally diagonalizable.*

Another consequence of Theorem 3 is the following result, which in particular allows the definition of the n-th root of a positive operator.

Corollary 4. *Let $T : \mathscr{H}^N \to \mathscr{H}^N$ be an invertible positive operator with normalized eigenvectors $(e_i)_{i=1}^N$ and respective eigenvalues $(\lambda_i)_{i=1}^N$, let $a \in \mathbb{R}$, and define an operator $T^a : \mathscr{H}^N \to \mathscr{H}^N$ by*

$$T^a x = \sum_{i=1}^N \lambda_i^a \langle x, e_i \rangle e_i \quad \textit{for all } x \in \mathscr{H}^N.$$

Then T^a is a positive operator and $T^a T^b = T^{a+b}$ for $a, b \in \mathbb{R}$. In particular, T^{-1} and $T^{-1/2}$ are positive operators.

Finally, we define the trace of an operator, which, by using Theorem 3, can be expressed in terms of eigenvalues.

Definition 12. Let $T : \mathscr{H}^N \to \mathscr{H}^N$ be an operator. Then, the *trace* of T is defined by

$$\operatorname{Tr} T = \sum_{i=1}^{N} \langle Te_i, e_i \rangle, \tag{1}$$

where $(e_i)_{i=1}^N$ is an arbitrary orthonormal basis for $\mathscr{H}^N$.

The trace is well defined since the sum in Equation 1 is independent of the choice of the orthonormal basis.

Corollary 5. *Let $T : \mathscr{H}^N \to \mathscr{H}^N$ be an orthogonally diagonalizable operator, and let $(\lambda_i)_{i=1}^N$ be its eigenvalues. Then*

$$\operatorname{Tr} T = \sum_{i=1}^{N} \lambda_i.$$

2.2.4 Projection Operators

Subspaces are closely intertwined with associated projection operators which map vectors onto the subspace either orthogonally or not. Although orthogonal projections are more often used, in Chapter [166] we will require the more general notion.

Definition 13. Let $P : \mathscr{H}^N \to \mathscr{H}^N$ be a linear operator. Then P is called a *projection*, if $P^2 = P$. This projection is called *orthogonal*, if P is in addition self-adjoint.

For the sake of brevity, *orthogonal projections* are often simply referred to as *projections* provided there is no danger of misinterpretation.

Relating to our previous comment, for any subspace $\mathscr{W}$ of $\mathscr{H}^N$, there exists a unique orthogonal projection P of $\mathscr{H}^N$ having $\mathscr{W}$ as its range. This projection can be constructed as follows: Let m denote the dimension of $\mathscr{W}$, and choose an orthonormal basis $(e_i)_{i=1}^m$ of $\mathscr{W}$. Then, for any $x \in \mathscr{H}^N$, we set

$$Px = \sum_{i=1}^{m} \langle x, e_i \rangle e_i.$$

It is important to notice that also $Id - P$ is an orthogonal projection of $\mathscr{H}^N$, this time onto the subspace $\mathscr{W}^\perp$.

An orthogonal projection P has the crucial property that each given vector of $\mathscr{H}^N$ is mapped to the closest vector in the range of P.

Lemma 1. *Let $\mathscr{W}$ be a subspace of $\mathscr{H}^N$, let P be the orthogonal projection onto $\mathscr{W}$, and let $x \in \mathscr{H}^N$. Then*

$$\|x - Px\| \leq \|x - \tilde{x}\| \quad \textit{for all } \tilde{x} \in \mathscr{W}.$$

Moreover, if $\|x - Px\| = \|x - \tilde{x}\|$ *for some* $\tilde{x} \in \mathscr{W}$, *then* $\tilde{x} = Px$.

The next result gives the relationship between trace and rank for projections. This follows from the definition of an orthogonal projection and Corollaries 3 and 5.

Proposition 8. *Let* P *be the orthogonal projection onto a subspace* $\mathscr{W}$ *of* $\mathscr{H}^N$, *and let* $m = \dim \mathscr{W}$. *Then* P *is orthogonally diagonalizable with eigenvalue* 1 *of multiplicity* m *and eigenvalue* 0 *of multiplicity* $N - m$. *In particular, we have that* $\mathrm{Tr}\, P = m$.

3 Basics of Finite Frame Theory

We start by presenting the basics of finite frame theory. For illustration purposes, we then present some exemplary frame classes. At this point, we would also like to refer to the monographs and books [35, 36, 100, 101, 112] as well as to [66, 67] for infinite-dimensional frame theory.

3.1 Definition of a Frame

The definition of a (Hilbert space) frame originates from early work by Duffin and Schaeffer [80] on nonharmonic Fourier series. The main idea, as already discussed in Section 1, is to weaken Parseval's identity yet to still retain norm equivalence between a signal and its frame coefficients.

Definition 14. A family of vectors $(\varphi_i)_{i=1}^M$ in $\mathscr{H}^N$ is called a *frame for* $\mathscr{H}^N$, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i=1}^{M} |\langle x, \varphi_i \rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathscr{H}^N. \tag{2}$$

The following notions are related to a frame $(\varphi_i)_{i=1}^M$.

(a) The constants A and B as in (2) are called *lower and upper frame bound* for the frame, respectively. The largest lower frame bound and the smallest upper frame bound are denoted by A_{op}, B_{op} and are called the *optimal frame bounds*.
(b) Any family $(\varphi_i)_{i=1}^M$ satisfying the right hand side inequality in (2) is called a *B-Bessel sequence*.
(c) If $A = B$ is possible in (2), then $(\varphi_i)_{i=1}^M$ is called an *A-tight frame*.
(d) If $A = B = 1$ is possible in (2) – i.e., Parseval's Identity holds –, then $(\varphi_i)_{i=1}^M$ is called a *Parseval frame*.
(e) If there exists a constant c such that $\|\varphi_i\| = c$ for all $i = 1, 2, \ldots, M$, then $(\varphi_i)_{i=1}^M$ is an *equal-norm frame*. If $c = 1$, $(\varphi_i)_{i=1}^M$ is a *unit-norm frame*.

(f) If there exists a constant c such that $|\langle \varphi_i, \varphi_j \rangle| = c$ for all $i \neq j$, then $(\varphi_i)_{i=1}^M$ is called an *equi-angular* frame.
(g) The values $(\langle x, \varphi_i \rangle)_{i=1}^M$ are called the *frame coefficients* of the vector x with respect to the frame $(\varphi_i)_{i=1}^M$.
(h) The frame $(\varphi_i)_{i=1}^M$ is called *exact*, if $(\varphi_i)_{i\in I}$ ceases to be a frame for $\mathscr{H}^N$ for every $I = \{1,\dots,M\} \setminus \{i_0\}$, $i_0 \in \{1,\dots,M\}$.

We can immediately make the following useful observations.

Lemma 2. *Let $(\varphi_i)_{i=1}^M$ be a family of vectors in $\mathscr{H}^N$.*

(i) If $(\varphi_i)_{i=1}^M$ is an orthonormal basis, then $(\varphi_i)_{i=1}^M$ is a Parseval frame. The converse is not true in general.
(ii) $(\varphi_i)_{i=1}^M$ is a frame for $\mathscr{H}^N$ if and only if it is a spanning set for $\mathscr{H}^N$.
(iii) $(\varphi_i)_{i=1}^M$ is a unit-norm Parseval frame if and only if it is an orthonormal basis.
(iv) If $(\varphi_i)_{i=1}^M$ is an exact frame for $\mathscr{H}^N$, then it is a basis of $\mathscr{H}^N$, i.e. a linearly independent spanning set.

Proof. (i). The first part is an immediate consequence of Proposition 1. For the second part, let $(e_i)_{i=1}^N$ and $(g_i)_{i=1}^N$ be orthonormal bases for $\mathscr{H}^N$. Then $(e_i/\sqrt{2})_{i=1}^N \cup (g_i/\sqrt{2})_{i=1}^N$ is a Parseval frame for $\mathscr{H}^N$, but not an orthonormal basis.

(ii). If $(\varphi_i)_{i=1}^M$ is not a spanning set for $\mathscr{H}^N$ then there exists $x \neq 0$ such that $\langle x, \varphi_i \rangle = 0$ for all $i = 1,\dots,M$. Hence, $(\varphi_i)_{i=1}^M$ cannot be a frame. Conversely, assume that $(\varphi_i)_{i=1}^M$ is not a frame. Then there exists a sequence $(x_n)_{n=1}^\infty$ of normalized vectors in $\mathscr{H}^N$ such that $\sum_{i=1}^M |\langle x_n, \varphi_i \rangle|^2 < 1/n$ for all $n \in \mathbb{N}$. Hence, the limit x of a convergent subsequence of $(x_n)_{n=1}^\infty$ satisfies $\langle x, \varphi_i \rangle = 0$ for all $i = 1,\dots,M$. Since $\|x\| = 1$, it follows that $(\varphi_i)_{i=1}^M$ is not a spanning set.

(iii). By the Parseval property, for each $i_0 \in \{1,\dots,M\}$, we have

$$\|\varphi_{i_0}\|_2^2 = \sum_{i=1}^M |\langle \varphi_{i_0}, \varphi_i \rangle|^2 = \|\varphi_{i_0}\|_2^4 + \sum_{i=1,\, i\neq i_0}^M |\langle \varphi_{i_0}, \varphi_i \rangle|^2.$$

Since the frame vectors are normalized, we conclude that

$$\sum_{i=1,\, i\neq i_0}^M |\langle \varphi_{i_0}, \varphi_i \rangle|^2 = 0 \quad \text{for all } i_0 \in \{1,\dots,M\}.$$

Hence $\langle \varphi_i, \varphi_j \rangle = 0$ for all $i \neq j$. Thus, $(\varphi_i)_{i=1}^M$ is an orthonormal system which is complete by (ii), and (iii) is proved.

(iv). If $(\varphi_i)_{i=1}^M$ is a frame, by (ii), it is also a spanning set for $\mathscr{H}^N$. Towards a contradiction, assume that $(\varphi_i)_{i=1}^M$ is linearly dependent. Then there exist some $i_0 \in \{1,\dots,M\}$ and values λ_i, $i \in I := \{1,\dots,M\} \setminus \{i_0\}$ such that

$$\varphi_{i_0} = \sum_{i\in I} \lambda_i \varphi_i.$$

This implies that $(\varphi_i)_{i\in I}$ is also a frame, thus contradicting exactness of the frame. □

Before presenting some insightful basic results in frame theory, we now first discuss some examples of frames to build up intuition.

3.2 Examples

By Lemma 2 (iii), orthonormal bases are unit-norm Parseval frames (and vice versa). However, applications typically require *redundant* Parseval frames. One basic way to approach this construction problem is to build redundant Parseval frames using orthonormal bases, and we will present several examples in the sequel. Since the associated proofs are straightforward, we leave them to the interested reader.

Example 1. Let $(e_i)_{i=1}^N$ be an orthonormal basis for $\mathscr{H}^N$.

(1) The system

$$(e_1, 0, e_2, 0, \ldots, e_N, 0)$$

is a Parseval frame for $\mathscr{H}^N$. This example indicates that a Parseval frame can indeed contain zero vectors.

(2) The system

$$\left(e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \ldots, \frac{e_N}{\sqrt{N}}, \ldots, \frac{e_N}{\sqrt{N}}\right),$$

is a Parseval frame for $\mathscr{H}^N$. This example indicates two important issues: Firstly, a Parseval frame can have multiple copies of a single vector. Secondly, the norms of vectors of an (infinite) Parseval frame can converge to zero.

We next consider a series of examples of non-Parseval frames.

Example 2. Let $(e_i)_{i=1}^N$ is an orthonormal basis for $\mathscr{H}^N$.

(1) The system

$$(e_1, e_1, \ldots, e_1, e_2, e_3, \ldots, e_N)$$

with the vector e_1 appearing $N+1$ times, is a frame for $\mathscr{H}^N$ with frame bounds 1 and $N+1$.

(2) The system

$$(e_1, e_1, e_2, e_2, e_3, e_3, \ldots, e_N)$$

is a 2-tight frame for $\mathscr{H}^N$.

(3) The union of L orthonormal bases of $\mathscr{H}^N$ is a unit-norm L-tight frame for $\mathscr{H}^N$, generalizing (2).

A particularly interesting example is the smallest truly redundant Parseval frame for $\mathbb{R}^2$, which is typically coined *Mercedes-Benz frame*. The reason for this naming becomes evident in Figure 1.

Example 3. The *Mercedes-Benz frame* for $\mathbb{R}^2$ is the equal-norm tight frame for $\mathbb{R}^2$ given by:

$$\left\{ \sqrt{\tfrac{2}{3}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \sqrt{\tfrac{2}{3}} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}, \sqrt{\tfrac{2}{3}} \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix} \right\}$$

Note that this frame is also equi-angular.

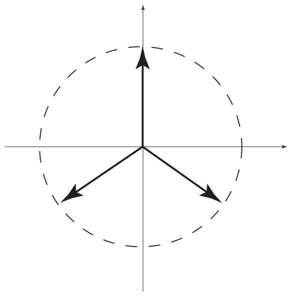

Fig. 1 Mercedes-Benz frame.

For more information on the theoretical aspects of equi-angular frames we refer to [61, 92, 121, 140]. A selection of their applications is reconstrucion without phase [6, 5], erasure-resilient transmission [16, 103], and coding [137]. We also refer to the chapters [157, 158] in this book.

Another standard class of examples can be derived from the *discrete Fourier transform* (*DFT*) *matrix*.

Example 4. Given $M \in \mathbb{N}$, we let $\omega = \exp(\frac{2\pi i}{M})$. Then the discrete Fourier transform (DFT) matrix in $\mathbb{C}^{M\times M}$ is defined by

$$D_M = \frac{1}{\sqrt{M}} \left(\omega^{jk}\right)_{j,k=0}^{M-1}.$$

This matrix is a unitary operator on $\mathbb{C}^M$. Later (see Corollary 11) it will be seen that the selection of any N rows from D_M, yields a Parseval frame for $\mathbb{C}^N$ by taking the associated M column vectors.

We would like to finally mention that Section 6 contains diverse constructions of frames. There also exist particularly interesting classes of frames such as Gabor frames utilized primarily for audio processing. Among the results on various aspects of Gabor frames are uncertainty considerations [114], linear independence [120], group-related properties [90], optimality analysis [128], and applications [68, 75, 76, 88, 89]. Chapter [159] provides a survey for this class of frames. Another example is the class of group frames, for which various constructions [25, 102, 148], classifications [65], and intriguing symmetry properties [147, 149] have been studied. A comprehensive presentation can be found in Chapter [158].

4 Frames and Operators

The analysis, synthesis, and frame operator determine the operation of a frame when analyzing and reconstructing a signal. The Grammian operator is perhaps not that well-known, yet it crucially illuminates the behavior of a frame $(\varphi_i)_{i=1}^M$ embedded as an N-dimensional subspace in the high-dimensional space $\mathbb{R}^M$.

For the rest of this introduction we set $\ell_2^M := \ell_2(\{1,\ldots,M\})$. Note that this space in fact coincides with $\mathbb{R}^M$ or $\mathbb{C}^M$, endowed with the standard inner product and the associated Euclidean norm.

4.1 Analysis and Synthesis Operators

Two of the main operators associated with a frame are the analysis and synthesis operators. The analysis operator – as the name suggests – analyzes a signal in terms of the frame by computing its frame coefficients. We start by formalizing this notion.

Definition 15. Let $(\varphi_i)_{i=1}^M$ be a family of vectors in $\mathscr{H}^N$. Then the associated *analysis operator* $T : \mathscr{H}^N \to \ell_2^M$ is defined by

$$Tx := \big(\langle x, \varphi_i\rangle\big)_{i=1}^M, \quad x \in \mathscr{H}^N.$$

In the following lemma we derive two basic properties of the analysis operator.

Lemma 3. *Let $(\varphi_i)_{i=1}^M$ be a sequence of vectors in $\mathscr{H}^N$ with associated analysis operator T.*

(i) We have

$$\|Tx\|^2 = \sum_{i=1}^M |\langle x, \varphi_i\rangle|^2 \quad \textit{for all } x \in \mathscr{H}^N.$$

Hence, $(\varphi_i)_{i=1}^M$ is a frame for $\mathscr{H}^N$ if and only if T is injective.

(ii) The adjoint operator $T^* : \ell_2^M \to \mathscr{H}^N$ of T is given by

$$T^*(a_i)_{i=1}^M = \sum_{i=1}^M a_i\varphi_i.$$

Proof. (i). This is an immediate consequence of the definition of T and the frame property (2).

(ii). For $x = (a_i)_{i=1}^M$ and $y \in \mathscr{H}^N$, we have

$$\langle T^*x, y\rangle = \langle x, Ty\rangle = \Big\langle (a_i)_{i=1}^M, \big(\langle y, \varphi_i\rangle\big)_{i=1}^M \Big\rangle = \sum_{i=1}^M a_i \overline{\langle y, \varphi_i\rangle} = \Big\langle \sum_{i=1}^M a_i\varphi_i, y \Big\rangle.$$

Thus, T^* is as claimed. □

The second main operator associated to a frame, the synthesis operator, is now defined as the adjoint operator to the analysis operator given in Lemma 3(ii).

Definition 16. Let $(\varphi_i)_{i=1}^M$ be a sequence of vectors in $\mathscr{H}^N$ with associated analysis operator T. Then the associated *synthesis operator* is defined to be the adjoint operator T^*.

The next result summarizes some basic, yet useful properties of the synthesis operator.

Lemma 4. *Let $(\varphi_i)_{i=1}^M$ be a sequence of vectors in $\mathscr{H}^N$ with associated analysis operator T.*

(i) Let $(e_i)_{i=1}^M$ denote the standard unit basis of ℓ_2^M. Then for all $i = 1,2,\dots,M$, we have $T^*e_i = T^*Pe_i = \varphi_i$, where $P : \ell_2^M \to \ell_2^M$ denotes the orthogonal projection onto $\operatorname{ran} T$.
(ii) $(\varphi_i)_{i=1}^M$ is a frame if and only if T^* is surjective.

Proof. The first claim follows immediately from Lemma 3 and the fact that $\ker T^* = (\operatorname{ran} T)^\perp$. The second claim is a consequence of $\operatorname{ran} T^* = (\ker T)^\perp$ and Lemma 3(i). □

Often frames are modified by the application of an invertible operator. The next result shows not only the impact on the associated analysis operator, but also the fact that the new sequence again forms a frame.

Proposition 9. *Let $\Phi = (\varphi_i)_{i=1}^M$ be a sequence of vectors in $\mathscr{H}^N$ with associated analysis operator T_Φ and let $F : \mathscr{H}^N \to \mathscr{H}^N$ be a linear operator. Then the analysis operator of the sequence $F\Phi = (F\varphi_i)_{i=1}^M$ is given by*

$$T_{F\Phi} = T_\Phi F^*.$$

Moreover, if Φ is a frame for $\mathscr{H}^N$ and F is invertible, then also $F\Phi$ is a frame for $\mathscr{H}^N$.

Proof. For $x \in \mathscr{H}^N$ we have

$$T_{F\Phi}x = \big(\langle x, F\varphi_i\rangle\big)_{i=1}^M = \big(\langle F^*x, \varphi_i\rangle\big)_{i=1}^M = T_\Phi F^* x.$$

This proves $T_{F\Phi} = T_\Phi F^*$. The *moreover*-part follows from Lemma 4(ii). □

Next, we analyze the structure of the matrix representation of the synthesis operator. This matrix is of fundamental importance, since this is what most frame constructions in fact focus on, see also Section 6.

The first result provides the form of this matrix alongside with stability properties.

Lemma 5. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with analysis operator T. Then a matrix representation of the synthesis operator T^* is the $N \times M$ matrix given by*

$$\begin{bmatrix} | & | & \cdots & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_M \\ | & | & \cdots & | \end{bmatrix}.$$

Moreover, the Riesz bounds of the row vectors of this matrix equal the frame bounds of the column vectors.

Proof. The form of the matrix representation is obvious. To prove the *moreover*-part, let $(e_j)_{j=1}^N$ be the corresponding orthonormal basis of $\mathscr{H}^N$ and for $j = 1,2,\ldots,N$ let

$$\psi_j = [\langle \varphi_1, e_j\rangle, \langle \varphi_2, e_j\rangle, \ldots, \langle \varphi_M, e_j\rangle],$$

be the row vectors of the matrix. Then for $x = \sum_{j=1}^N a_j e_j$ we obtain

$$\begin{aligned}\sum_{i=1}^M |\langle x, \varphi_i\rangle|^2 &= \sum_{i=1}^M \left| \sum_{j=1}^N a_j \langle e_j, \varphi_i\rangle \right|^2 = \sum_{j,k=1}^N a_j \overline{a_k} \sum_{i=1}^M \langle e_j, \varphi_i\rangle \langle \varphi_i, e_k\rangle \\ &= \sum_{j,k=1}^N a_j \overline{a_k} \langle \psi_k, \psi_j\rangle = \left\| \sum_{j=1}^N \overline{a_j} \psi_j \right\|^2.\end{aligned}$$

The claim follows from here. □

A much stronger result (Proposition 12) can be proven for the case in which the matrix representation is derived using a specifically chosen orthonormal basis. The choice of this orthonormal basis though requires the introduction of the so-called frame operator in the following Subsection 4.2.

4.2 The Frame Operator

The frame operator might be considered the most important operator associated with a frame. Although it is 'merely' the concatenation of the analysis and synthesis operator, it encodes crucial properties of the frame as we will see in the sequel. Moreover, it is also fundamental for the reconstruction of signals from frame coefficients (see Theorem 8).

4.2.1 Fundamental Properties

The precise definition of the frame operator associated with a frame is as follows.

Definition 17. Let $(\varphi_i)_{i=1}^M$ be a sequence of vectors in $\mathscr{H}^N$ with associated analysis operator T. Then the associated *frame operator* $S : \mathscr{H}^N \to \mathscr{H}^N$ is defined by

$$Sx := T^*Tx = \sum_{i=1}^{M} \langle x, \varphi_i \rangle \varphi_i, \quad x \in \mathscr{H}^N.$$

A first observation concerning the close relation of the frame operator to frame properties is the following lemma.

Lemma 6. *Let $(\varphi_i)_{i=1}^M$ be a sequence of vectors in $\mathscr{H}^N$ with associated frame operator S. Then, for all $x \in \mathscr{H}^N$,*

$$\langle Sx, x \rangle = \sum_{i=1}^{M} |\langle x, \varphi_i \rangle|^2.$$

Proof. This follows directly from $\langle Sx, x \rangle = \langle T^*Tx, x \rangle = \|Tx\|^2$ and Lemma 3(i). □

Clearly, the frame operator $S = T^*T$ is self-adjoint and positive. The most fundamental property of the frame operator – if the underlying sequence of vectors forms a frame – is its invertibility which is crucial for the reconstruction formula.

Theorem 4. *The frame operator S of a frame $(\varphi_i)_{i=1}^M$ for $\mathscr{H}^N$ with frame bounds A and B is a positive, self-adjoint invertible operator satisfying*

$$A \cdot Id \leq S \leq B \cdot Id.$$

Proof. By Lemma 6, we have

$$\langle Ax, x \rangle = A\|x\|^2 \leq \sum_{i=1}^{M} |\langle x, \varphi_i \rangle|^2 = \langle Sx, x \rangle \leq B\|x\|^2 = \langle Bx, x \rangle \quad \text{for all } x \in \mathscr{H}^N.$$

This implies the claimed inequality. □

The following proposition follows directly from Proposition 9.

Proposition 10. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with frame operator S, and let F be an invertible operator on $\mathscr{H}^N$. Then $(F\varphi_i)_{i=1}^M$ is a frame with frame operator FSF^*.*

4.2.2 The Special Case of Tight Frames

Tight frames can be characterized as those frames whose frame operator equals a positive multiple of the identity. The next result provides a variety of similarly frame operator inspired classifications.

Proposition 11. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with analysis operator T and frame operator S. Then the following conditions are equivalent.*

(i) $(\varphi_i)_{i=1}^M$ *is an A-tight frame for $\mathscr{H}^N$.*
(ii) $S = A \cdot Id$.

(iii) For every $x \in \mathscr{H}^N$,

$$x = A^{-1} \cdot \sum_{i=1}^{M} \langle x, \varphi_i \rangle \varphi_i.$$

(iv) For every $x \in \mathscr{H}^N$,

$$A\|x\|^2 = \sum_{i=1}^{M} |\langle x, \varphi_i \rangle|^2.$$

(v) $T/\sqrt{A}$ is an isometry.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). These are immediate from the definition of the frame operator and from Theorem 4.

(ii) $\Leftrightarrow$ (v). This follows from the fact that $T/\sqrt{A}$ is an isometry if and only if $T^*T = A \cdot Id$. $\square$

A similar result for the special case of a Parseval frame can be easily deduced from Proposition 11 by setting $A = 1$.

4.2.3 Eigenvalues of the Frame Operator

Tight frames have the property that the eigenvalues of the associated frame operator all coincide. We next consider the general situation, i.e., frame operators with arbitrary eigenvalues.

The first and maybe even most important result shows that the largest and smallest eigenvalues of the frame operator are the optimal frame bounds of the frame. Optimality refers to the smallest upper frame bound and the largest lower frame bound.

Theorem 5. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with frame operator S having eigenvalues $\lambda_1 \geq \ldots \geq \lambda_N$. Then λ_1 coincides with the optimal upper frame bound and λ_N is the optimal lower frame bound.*

Proof. Let $(e_i)_{i=1}^N$ denote the normalized eigenvectors of the frame operator S with respective eigenvalues $(\lambda_j)_{j=1}^N$ written in decreasing order. Let $x \in \mathscr{H}^N$ be arbitrarily fixed. Since $x = \sum_{j=1}^M \langle x, e_j \rangle e_j$, we obtain

$$Sx = \sum_{j=1}^{N} \lambda_j \langle x, e_j \rangle e_j.$$

By Lemma 6, this implies

$$\begin{aligned} \sum_{i=1}^{M} |\langle x, \varphi_i \rangle|^2 &= \langle Sx, x \rangle = \left\langle \sum_{j=1}^{N} \lambda_j \langle x, e_j \rangle e_j, \sum_{j=1}^{N} \langle x, e_j \rangle e_j \right\rangle \\ &= \sum_{j=1}^{N} \lambda_j |\langle x, e_j \rangle|^2 \leq \lambda_1 \sum_{j=1}^{N} |\langle x, e_j \rangle|^2 = \lambda_1 \|x\|^2. \end{aligned}$$

Thus $B_{op} \leq \lambda_1$, where B_{op} denotes the optimal upper frame bound of the frame $(\varphi_i)_{i=1}^M$. The claim $B_{op} = \lambda_1$ then follows from

$$\sum_{i=1}^M |\langle e_1, \varphi_i \rangle|^2 = \langle Se_1, e_1 \rangle = \langle \lambda_1 e_1, e_1 \rangle = \lambda_1.$$

The claim concerning the lower frame bound can be proven similarly. □

From this result, we can now draw the following immediate conclusion on Riesz bounds.

Corollary 6. *Let $(\varphi_i)_{i=1}^N$ be a frame for $\mathscr{H}^N$. Then the following statements hold.*

(i) The optimal upper Riesz bound and the optimal upper frame bound of $(\varphi_i)_{i=1}^N$ coincide.
(ii) The optimal lower Riesz bound and the optimal lower frame bound of $(\varphi_i)_{i=1}^N$ coincide.

Proof. Let T denote the analysis operator of $(\varphi_i)_{i=1}^N$ and S the associated frame operator having eigenvalues $(\lambda_i)_{i=1}^N$ written in decreasing order. We have

$$\lambda_1 = \|S\| = \|T^*T\| = \|T\|^2 = \|T^*\|^2$$

and

$$\lambda_N = \|S^{-1}\|^{-1} = \|(T^*T)^{-1}\|^{-1} = \|(T^*)^{-1}\|^{-2}.$$

Now, both claims follow from Theorem 5, Lemma 4, and Proposition 5. □

The next theorem reveals a relation between the frame vectors and the eigenvalues and eigenvectors of the associated frame operator.

Theorem 6. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with frame operator S having normalized eigenvectors $(e_j)_{j=1}^N$ and respective eigenvalues $(\lambda_j)_{j=1}^N$. Then for all $j = 1, 2, \ldots, N$ we have*

$$\lambda_j = \sum_{i=1}^M |\langle e_j, \varphi_i \rangle|^2.$$

In particular,

$$\operatorname{Tr} S = \sum_{j=1}^N \lambda_j = \sum_{i=1}^M \|\varphi_i\|^2.$$

Proof. This follows from $\lambda_j = \langle Se_j, e_j \rangle$ for all $j = 1, \ldots, N$ and Lemma 6. □

4.2.4 Structure of the Synthesis Matrix

As already promised in Subsection 4.1, we now apply the previously derived results to obtain a complete characterization of the synthesis matrix of a frame in terms of the frame operator.

Proposition 12. *Let $T:\mathscr{H}^N \to \ell_2^M$ be a linear operator, let $(e_j)_{j=1}^N$ be an orthonormal basis of $\mathscr{H}^N$ and let $(\lambda_j)_{j=1}^N$ be a sequence of positive numbers. By A denote the $N \times M$ matrix representation of T^* with respect to $(e_j)_{j=1}^N$ (and the standard unit basis $(\hat{e}_i)_{i=1}^M$ of ℓ_2^M). Then the following conditions are equivalent.*

(i) $(T^*\hat{e}_i)_{i=1}^M$ forms a frame for $\mathscr{H}^N$ whose frame operator has eigenvectors $(e_j)_{j=1}^N$ and associated eigenvalues $(\lambda_j)_{j=1}^N$.
(ii) The rows of A are orthogonal, and the j-th row square sums to λ_j.
(iii) The columns of A form a frame for ℓ_2^N, and $AA^* = \mathrm{diag}(\lambda_1,\dots,\lambda_N)$.

Proof. Let $(f_j)_{j=1}^N$ be the standard unit basis of ℓ_2^N and denote by $U:\ell_2^N \to \mathscr{H}^N$ the unitary operator which maps f_j to e_j. Then $T^* = UA$.

(i)⇒(ii). For $j,k \in \{1,\dots,N\}$ we have

$$\langle A^* f_j, A^* f_k\rangle = \langle TU f_j, TU f_k\rangle = \langle T^*Te_j, e_k\rangle = \lambda_j \delta_{jk},$$

which is equivalent to (ii).

(ii)⇒(iii). Since the rows of A are orthogonal, we have $\mathrm{rank}A = N$ which implies that the columns of A form a frame for ℓ_2^N. The rest follows from $\langle AA^* f_j, f_k\rangle = \langle A^* f_j, A^* f_k\rangle = \lambda_j \delta_{jk}$ for $j,k = 1,\dots,N$.

(iii)⇒(i). Since $(A\hat{e}_i)_{i=1}^M$ is a spanning set for ℓ_2^N and $T^* = UA$, it follows that $(T^*\hat{e}_i)_{i=1}^M$ forms a frame for $\mathscr{H}^N$. Its analysis operator is given by T since for all $x \in \mathscr{H}^N$,

$$(\langle x, T^*\hat{e}_i\rangle)_{i=1}^M = (\langle Tx, \hat{e}_i\rangle)_{i=1}^M = Tx.$$

Moreover,

$$T^*Te_j = UAA^*U^*e_j = U\,\mathrm{diag}(\lambda_1,\dots,\lambda_N)\,f_j = \lambda_j U f_j = \lambda_j e_j,$$

which completes the proof. □

4.3 Grammian Operator

Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with analysis operator T. The previous subsection was concerned with properties of the frame operator defined by $S = T^*T : \mathscr{H}^N \to \mathscr{H}^N$. Of particular interest is also the operator generated by first applying the synthesis and then the analysis operator. Let us first state the precise definition before discussing its importance.

Definition 18. Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with analysis operator T. Then the operator $G:\ell_2^M \to \ell_2^M$ defined by

$$G(a_i)_{i=1}^M = TT^*(a_i)_{i=1}^M = \left(\sum_{i=1}^M a_i \langle \varphi_i, \varphi_k\rangle\right)_{k=1}^M = \sum_{i=1}^M a_i (\langle \varphi_i, \varphi_k\rangle)_{k=1}^M$$

is called the *Grammian* (*operator*) of the frame $(\varphi_i)_{i=1}^M$.

Note that the (canonical) matrix representation of the Grammian of a frame $(\varphi_i)_{i=1}^M$ for $\mathscr{H}^N$ (which will also be called the *Grammian matrix*) is given by

$$\begin{bmatrix} \|\varphi_1\|^2 & \langle\varphi_2,\varphi_1\rangle & \cdots & \langle\varphi_M,\varphi_1\rangle \\ \langle\varphi_1,\varphi_2\rangle & \|\varphi_2\|^2 & \cdots & \langle\varphi_M,\varphi_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\varphi_1,\varphi_M\rangle & \langle\varphi_2,\varphi_M\rangle & \cdots & \|\varphi_M\|^2 \end{bmatrix}.$$

One property of the Grammian is immediate. In fact, if the frame is unit-norm then the entries of the Grammian matrix are exactly the cosines of the angles between the frame vectors. Hence, for instance, if a frame is equi-angular then all off diagonal entries of the Grammian matrix have the same modulus.

The fundamental properties of the Grammian operator are collected in the following result.

Theorem 7. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with analysis operator T, frame operator S, and Grammian operator G. Then the following statements hold.*

(i) An operator U on $\mathscr{H}^N$ is unitary if and only if the Grammian of $(U\varphi_i)_{i=1}^M$ coincides with G.
(ii) The non-zero eigenvalues of G and S coincide.
(iii) $(\varphi_i)_{i=1}^M$ is a Parseval frame if and only if G is an orthogonal projection of rank N (namely onto the range of T).
(iv) G is invertible if and only if $M = N$.

Proof. (i). This follows immediately from the fact that the entries of the Grammian matrix for $(U\varphi_i)_{i=1}^M$ are of the form $\langle U\varphi_i, U\varphi_j\rangle$.

(ii). Since TT^* and T^*T have the same non-zero eigenvalues (see Proposition 7), the same is true for G and S.

(iii). It is immediate to prove that G is self-adjoint and has rank N. Since T is injective, T^* is surjective, and

$$G^2 = (TT^*)(TT^*) = T(T^*T)T^*,$$

it follows that G is an orthogonal projection if and only if $T^*T = Id$, which is equivalent to the frame being Parseval.

(iv). This is immediate by (ii). □

5 Reconstruction from Frame Coefficients

The analysis of a signal is typically performed by merely considering its frame coefficients. However, if the task is transmission of a signal, the ability to reconstruct the signal from its frame coefficients and also to do so efficiently becomes crucial.

Reconstruction from coefficients with respect to an orthonormal basis was already discussed in Corollary 1. However, reconstruction from coefficients with respect to a redundant system is much more delicate and requires the utilization of another frame, called dual frame. If computing such a dual frame is computationally too complex, a circumvention of this problem is the so-called frame algorithm.

5.1 Exact Reconstruction

We start with stating an exact reconstruction formula.

Theorem 8. *Let* $(\varphi_i)_{i=1}^M$ *be a frame for* $\mathscr{H}^N$ *with frame operator* S*. Then, for every* $x \in \mathscr{H}^N$*, we have*

$$x = \sum_{i=1}^M \langle x, \varphi_i \rangle S^{-1}\varphi_i = \sum_{i=1}^M \langle x, S^{-1}\varphi_i \rangle \varphi_i.$$

Proof. This follows directly from the definition of the frame operator in Definition 17 by writing $x = S^{-1}Sx$ and $x = SS^{-1}x$. □

Notice that the first formula can be interpreted as a reconstruction strategy, whereas the second formula has the flavor of a decomposition. We further observe that the sequence $(S^{-1}\varphi_i)_{i=1}^M$ plays a crucial role in the formulas in Theorem 8. The next result shows that this sequence indeed also constitutes a frame.

Proposition 13. *Let* $(\varphi_i)_{i=1}^M$ *be a frame for* $\mathscr{H}^N$ *with frame bounds* A *and* B *and with frame operator* S*. Then the sequence* $(S^{-1}\varphi_i)_{i=1}^M$ *is a frame for* $\mathscr{H}^N$ *with frame bounds* B^{-1} *and* A^{-1} *and with frame operator* S^{-1}*.*

Proof. By Proposition 10, the sequence $(S^{-1}\varphi_i)_{i=1}^M$ forms a frame for $\mathscr{H}^N$ with associated frame operator $S^{-1}S(S^{-1})^* = S^{-1}$. This in turn yields the frame bounds B^{-1} and A^{-1}. □

This new frame is called the *canonical dual frame*. In the sequel, we will discuss that also other dual frames may be utilized for reconstruction.

Definition 19. Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with frame operator denoted by S. Then $(S^{-1}\varphi_i)_{i=1}^M$ is called the *canonical dual frame* for $(\varphi_i)_{i=1}^M$.

The canonical dual frame of a Parseval frame is now easily determined by Proposition 13.

Corollary 7. *Let* $(\varphi_i)_{i=1}^M$ *be a Parseval frame for* $\mathscr{H}^N$*. Then its canonical dual frame is the frame* $(\varphi_i)_{i=1}^M$ *itself, and the reconstruction formula in Theorem 8 reads*

$$x = \sum_{i=1}^M \langle x, \varphi_i \rangle \varphi_i, \quad x \in \mathscr{H}^N.$$

As an application of the above reconstruction formula for Parseval frames, we prove the following proposition which again shows the close relation between Parseval frames and orthonormal bases already indicated in Lemma 2.

Proposition 14 (Trace Formula for Parseval Frames). *Let $(\varphi_i)_{i=1}^M$ be a Parseval frame for $\mathscr{H}^N$, and let F be a linear operator on $\mathscr{H}^N$. Then*

$$\mathrm{Tr}(F) = \sum_{i=1}^M \langle F\varphi_i, \varphi_i \rangle.$$

Proof. Let $(e_j)_{j=1}^N$ be an orthonormal basis for $\mathscr{H}^N$. Then, by definition,

$$\mathrm{Tr}(F) = \sum_{j=1}^N \langle Fe_j, e_j \rangle.$$

This implies

$$\begin{aligned}\mathrm{Tr}(F) &= \sum_{j=1}^N \left\langle \sum_{i=1}^M \langle Fe_j, \varphi_i \rangle \varphi_i, e_j \right\rangle = \sum_{j=1}^N \sum_{i=1}^M \langle e_j, F^*\varphi_i \rangle \langle \varphi_i, e_j \rangle \\ &= \sum_{i=1}^M \left\langle \sum_{j=1}^N \langle \varphi_i, e_j \rangle e_j, F^*\varphi_i \right\rangle = \sum_{i=1}^M \langle \varphi_i, F^*\varphi_i \rangle = \sum_{i=1}^M \langle F\varphi_i, \varphi_i \rangle. \quad \square\end{aligned}$$

As already announced, many other dual frames for reconstruction exist. We next provide a precise definition.

Definition 20. Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$. Then a frame $(\psi_i)_{i=1}^M$ is called a *dual frame* for $(\varphi_i)_{i=1}^M$, if

$$x = \sum_{i=1}^M \langle x, \varphi_i \rangle \psi_i \quad \text{for all } x \in \mathscr{H}^N.$$

Dual frames, which do not coincide with the canonical dual frame, are often coined *alternate dual frames*.

Similar to the different forms of the reconstruction formula in Theorem 8, also dual frames can achieve reconstruction in different ways.

Proposition 15. *Let $(\varphi_i)_{i=1}^M$ and $(\psi_i)_{i=1}^M$ be frames for $\mathscr{H}^N$ and let T and $\tilde{T}$ be the analysis operators of $(\varphi_i)_{i=1}^M$ and $(\psi_i)_{i=1}^M$, respectively. Then the following conditions are equivalent.*

(i) We have $x = \sum_{i=1}^M \langle x, \psi_i \rangle \varphi_i$ for all $x \in \mathscr{H}^N$.
(ii) We have $x = \sum_{i=1}^M \langle x, \varphi_i \rangle \psi_i$ for all $x \in \mathscr{H}^N$.
(iii) We have $\langle x, y \rangle = \sum_{i=1}^M \langle x, \varphi_i \rangle \langle \psi_i, y \rangle$ for all $x, y \in \mathscr{H}^N$.
(iv) $T^*\tilde{T} = Id$ and $\tilde{T}^*T = Id$.

Proof. Clearly (i) is equivalent to $T^*\tilde{T} = Id$ which holds if and only if $\tilde{T}^*T = Id$. The equivalence of (iii) can be derived in a similar way. □

One might ask what distinguishes the canonical dual frame from the alternate dual frames besides its explicit formula in terms of the initial frame. Another seemingly different question is which properties of the coefficient sequence in the decomposition of some signal x in terms of the frame (see Theorem 8),

$$x = \sum_{i=1}^{M} \langle x, S^{-1}\varphi_i \rangle \varphi_i,$$

uniquely distinguishes it from other coefficient sequences – redundancy allows infinitely many coefficient sequences. Interestingly, the next result answers both questions simultaneously by stating that this coefficient sequence has minimal ℓ_2-norm among all sequences – in particular those, with respect to alternate dual frames – representing x.

Proposition 16. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with frame operator S, and let $x \in \mathscr{H}^N$. If $(a_i)_{i=1}^M$ are scalars such that $x = \sum_{i=1}^M a_i \varphi_i$, then*

$$\sum_{i=1}^{M} |a_i|^2 = \sum_{i=1}^{M} |\langle x, S^{-1}\varphi_i \rangle|^2 + \sum_{i=1}^{M} |a_i - \langle x, S^{-1}\varphi_i \rangle|^2.$$

Proof. Letting T denote the analysis operator of $(\varphi_i)_{i=1}^M$, we obtain

$$(\langle x, S^{-1}\varphi_i \rangle)_{i=1}^M = (\langle S^{-1}x, \varphi_i \rangle)_{i=1}^M \in \operatorname{ran} T.$$

Since $x = \sum_{i=1}^M a_i \varphi_i$, it follows that

$$(a_i - \langle x, S^{-1}\varphi_i \rangle)_{i=1}^M \in \ker T^* = (\operatorname{ran} T)^{\perp}.$$

Considering the decomposition

$$(a_i)_{i=1}^M = (\langle x, S^{-1}\varphi_i \rangle)_{i=1}^M + (a_i - \langle x, S^{-1}\varphi_i \rangle)_{i=1}^M,$$

the claim is immediate. □

Corollary 8. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$, and let $(\psi_i)_{i=1}^M$ be an associated alternate dual frame. Then, for all $x \in \mathscr{H}^N$,*

$$\|(\langle x, S^{-1}\varphi_i \rangle)_{i=1}^M\|_2 \le \|(\langle x, \psi_i \rangle)_{i=1}^M\|_2.$$

We wish to mention that also sequences which are minimal in the ℓ_1 norm play a crucial role to date due to the fact that the ℓ_1 norm promotes sparsity. The interested reader is referred to Chapter [162] for further details.

5.2 Properties of Dual Frames

While focussing on properties of the canonical dual frame in the last subsection, we next discuss properties shared by all dual frames. The first question arising is: How do you characterize all dual frames? A comprehensive answer is provided by the following result.

Proposition 17. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with analysis operator T and frame operator S. Then the following conditions are equivalent.*

(i) $(\psi_i)_{i=1}^M$ is a dual frame for $(\varphi_i)_{i=1}^M$.
(ii) The analysis operator T_1 of the sequence $(\psi_i - S^{-1}\varphi_i)_{i=1}^M$ satisfies

$$\operatorname{ran} T \perp \operatorname{ran} T_1.$$

Proof. We set $\tilde{\varphi}_i := \psi_i - S^{-1}\varphi_i$ for $i = 1, \ldots, M$ and note that

$$\sum_{i=1}^M \langle x, \psi_i \rangle \varphi_i = \sum_{i=1}^M \langle x, \tilde{\varphi}_i + S^{-1}\varphi_i \rangle \varphi_i = x + \sum_{i=1}^M \langle x, \tilde{\varphi}_i \rangle \varphi_i = x + T^* T_1 x$$

holds for all $x \in \mathscr{H}^N$. Hence, $(\psi_i)_{i=1}^M$ is a dual frame for $(\varphi_i)_{i=1}^M$ if and only if $T^* T_1 = 0$. But this is equivalent to (ii). $\square$

From this result, we have the following corollary which provides a general formula for all dual frames.

Corollary 9. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with analysis operator T and frame operator S with associated normalized eigenvectors $(e_j)_{j=1}^N$ and respective eigenvalues $(\lambda_j)_{j=1}^N$. Then every dual frame $\{\psi_i\}_{i=1}^M$ for $(\varphi_i)_{i=1}^M$ is of the form*

$$\psi_i = \sum_{j=1}^N \left(\frac{1}{\lambda_j} \langle \varphi_i, e_j \rangle + \overline{h_{ij}} \right) e_j, \quad i = 1, \ldots, M,$$

where each $(h_{ij})_{i=1}^M$, $j = 1, \ldots, N$, is an element of $(\operatorname{ran} T)^\perp$.

Proof. If ψ_i, $i = 1, \ldots, M$, is of the given form with sequences $(h_{ij})_{i=1}^M \in \ell_2^M$, $j = 1, \ldots, N$, then $\psi_i = S^{-1}\varphi_i + \tilde{\varphi}_i$, where $\tilde{\varphi}_i := \sum_{j=1}^N \overline{h_{ij}} e_j$, $i = 1, \ldots, M$. The analysis operator $\tilde{T}$ of $(\tilde{\varphi}_i)_{i=1}^M$ satisfies $\tilde{T} e_j = (h_{ij})_{i=1}^M$. The claim follows from this observation. $\square$

As a second corollary, we derive a characterization of all frames which have a uniquely determined dual frame. It is evident, that this unique dual frame coincides with the canonical dual frame.

Corollary 10. *A frame $(\varphi_i)_{i=1}^M$ for $\mathscr{H}^N$ has a unique dual frame if and only if $M = N$.*

5.3 Frame Algorithms

Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with frame operator S, and assume we are given the image of a signal $x \in \mathscr{H}^N$ under the analysis operator, i.e., the sequence $(\langle x, \varphi_i \rangle)_{i=1}^M$ in ℓ_2^M. Theorem 8 already provided us with the reconstruction formula

$$x = \sum_{i=1}^{M} \langle x, \varphi_i \rangle S^{-1} \varphi_i$$

by using the canonical dual frame. Since inversion is typically not only computationally expensive, but also numerically instable, this formula might not be utilizable in practice.

To resolve this problem, we will next discuss three iterative methods to derive a converging sequence of approximations of x from knowledge of $(\langle x, \varphi_i \rangle)_{i=1}^M$. The first on our list is the so-called *frame algorithm*.

Proposition 18 (Frame Algorithm). *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with frame bounds A, B and frame operator S. Given a signal $x \in \mathscr{H}^N$, define a sequence $(y_j)_{j=0}^\infty$ in $\mathscr{H}^N$ by*

$$y_0 = 0, \quad y_j = y_{j-1} + \frac{2}{A+B} S(x - y_{j-1}) \ \text{ for all } j \geq 1.$$

Then $(y_j)_{j=0}^\infty$ converges to x in $\mathscr{H}^N$ and the rate of convergence is

$$\|x - y_j\| \leq \left(\frac{B-A}{B+A} \right)^j \|x\|, \quad j \geq 0.$$

Proof. First, for all $x \in \mathscr{H}^N$, we have

$$\left\langle \left(Id - \frac{2}{A+B} S \right) x, x \right\rangle = \|x\|^2 - \frac{2}{A+B} \sum_{i=1}^{M} |\langle x, \varphi_i \rangle|^2 \leq \|x\|^2 - \frac{2A}{A+B} \|x\|^2 = \frac{B-A}{A+B} \|x\|^2.$$

Similarly, we obtain

$$-\frac{B-A}{B+A} \|x\|^2 \leq \left\langle \left(Id - \frac{2}{A+B} S \right) x, x \right\rangle,$$

which yields

$$\left\| Id - \frac{2}{A+B} S \right\| \leq \frac{B-A}{A+B}. \tag{3}$$

By the definition of y_j, for any $j \geq 0$,

$$x - y_j = x - y_{j-1} - \frac{2}{A+B} S(x - y_{j-1}) = \left(Id - \frac{2}{A+B} S \right)(x - y_{j-1}).$$

Iterating this calculation, we derive

$$x - y_j = \left(Id - \frac{2}{A+B}S\right)^j (x - y_0), \quad \text{for all } j \geq 0.$$

Thus, by (3),

$$\begin{aligned} \|x - y_j\| &= \left\| \left(Id - \frac{2}{A+B}S\right)^j (x - y_0) \right\| \\ &\leq \left\| Id - \frac{2}{A+B}S \right\|^j \|x - y_0\| \\ &\leq \left(\frac{B-A}{A+B}\right)^j \|x\|. \end{aligned}$$

The result is proved. □

We wish to mention that, although the iteration formula in the frame algorithm contains x, the algorithm does not depend on the knowledge of x but only on the frame coefficients $(\langle x, \varphi_i \rangle)_{i=1}^M$, since $y_j = y_{j-1} + \frac{2}{A+B}(\sum_i \langle x, \varphi_i \rangle \varphi_i - S y_{j-1})$.

One drawback of the frame algorithm is the fact that not only does the convergence rate depend on the ratio of the frame bounds, i.e., the condition number of the frame, but it depends on it in a highly sensitive way. This causes the problem that a large ratio of the frame bounds leads to very slow convergence.

To tackle this problem, in [97], the *Chebyshev method* and the *conjugate gradient methods* were introduced, which are significantly better adapted to frame theory leading to faster convergence than the frame algorithm. These two algorithms will next be discussed. We start with the *Chebyshev algorithm*.

Proposition 19 (Chebychev Algorithm, [97]). *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathcal{H}^N$ with frame bounds A, B and frame operator S, and set*

$$\rho := \frac{B-A}{B+A} \quad \textit{and} \quad \sigma := \frac{\sqrt{B} - \sqrt{A}}{\sqrt{B} + \sqrt{A}}.$$

Given a signal $x \in \mathcal{H}^N$, define a sequence $(y_j)_{j=0}^\infty$ in $\mathcal{H}^N$ and corresponding scalars $(\lambda_j)_{j=1}^\infty$ by

$$y_0 = 0, \quad y_1 = \frac{2}{B+A} Sx, \quad \textit{and} \quad \lambda_1 = 2,$$

and for $j \geq 2$, set

$$\lambda_j = \frac{1}{1 - \frac{\rho^2}{4}\lambda_{j-1}} \quad \textit{and} \quad y_j = \lambda_j \left(y_{j-1} - y_{j-2} + \frac{2}{B+A} S\left(x - y_{j-1}\right)\right) + y_{j-2}.$$

Then $(y_j)_{j=0}^\infty$ converges to x in $\mathcal{H}^N$ and the rate of convergence is

$$\|x - y_j\| \leq \frac{2\sigma^j}{1 + \sigma^{2j}} \|x\|.$$

The advantage of the *conjugate gradient method*, which we will present next, is the fact that it does not require knowledge of the frame bounds. However, as before, the rate of convergence certainly does depend on them.

Proposition 20 (Conjugate Gradient Method, [97]). *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with frame operator S. Given a signal $x \in \mathscr{H}^N$, define three sequences $(y_j)_{j=0}^\infty$, $(r_j)_{j=0}^\infty$, and $(p_j)_{j=-1}^\infty$ in $\mathscr{H}^N$ and corresponding scalars $(\lambda_j)_{j=-1}^\infty$ by*

$$y_0 = 0, \quad r_0 = p_0 = Sx, \quad \textit{and} \quad p_{-1} = 0,$$

and for $j \geq 0$, set

$$\lambda_j = \frac{\langle r_j, p_j \rangle}{\langle p_j, Sp_j \rangle}, \quad y_{j+1} = y_j + \lambda_j p_j, \quad r_{j+1} = r_j - \lambda_j S p_j,$$

and

$$p_{j+1} = Sp_j - \frac{\langle Sp_j, Sp_j \rangle}{\langle p_j, Sp_j \rangle} p_j - \frac{\langle Sp_j, Sp_{j-1} \rangle}{\langle p_{j-1}, Sp_{j-1} \rangle} p_{j-1}.$$

Then $(y_j)_{j=0}^\infty$ converges to x in $\mathscr{H}^N$ and the rate of convergence is

$$|||x - y_j||| \leq \frac{2\sigma^j}{1+\sigma^{2j}} |||x||| \quad \textit{with} \quad \sigma = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}},$$

and $|||\cdot|||$ is the norm on $\mathscr{H}^N$ given by $|||x||| = \langle x, Sx \rangle^{1/2} = \|S^{1/2}x\|$, $x \in \mathscr{H}^N$.

6 Construction of Frames

As diverse as the various desiderata are which applications require a frame to satisfy, as diverse are also the methods to construct frames [37, 59]. In this section, we will present a prominent selection. For further details and results, such as, for example, the construction of frames through spectral tetris [31, 47, 44] and through eigensteps [30], we refer to Chapter [155].

6.1 Tight and Parseval Frames

Tight frames are particularly desirable due to the fact that the reconstruction of a signal from tight frame coefficients is numerically optimally stable as already discussed in Section 5. Most of the constructions we will present utilize a given frame, which is then modified to become a tight frame.

We start with the most basic result for generating a Parseval frame which is the application of $S^{-1/2}$, S being the frame operator.

Lemma 7. *If $(\varphi_i)_{i=1}^M$ is a frame for $\mathscr{H}^N$ with frame operator S, then $(S^{-1/2}\varphi_i)_{i=1}^M$ is a Parseval frame.*

Proof. By Proposition 10, the frame operator for $(S^{-1/2}\varphi_i)_{i=1}^M$ is $S^{-1/2}SS^{-1/2} = Id$. □

Although this result is impressive in its simplicity, from a practical point of view it has various problems, the most significant being that this procedure requires inversion of the frame operator.

However, Lemma 7 can certainly be applied if all eigenvalues and respective eigenvectors of the frame operator are given. If only information on the eigenspace corresponding to the largest eigenvalue is missing, then there exists a simple practical method to generate a tight frame by adding a provably minimal number of vectors.

Proposition 21. *Let $(\varphi_i)_{i=1}^M$ be any family of vectors in $\mathscr{H}^N$ with frame operator S having eigenvectors $(e_j)_{j=1}^N$ and respective eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. Let $1 \leq k \leq N$ be such that $\lambda_1 = \lambda_2 = \ldots = \lambda_k > \lambda_{k+1}$. Then*

$$(\varphi_i)_{i=1}^M \cup \Big((\lambda_1 - \lambda_j)^{1/2} e_j\Big)_{j=k+1}^N \tag{4}$$

forms a λ_1-tight frame for $\mathscr{H}^N$.

Moreover, $N-k$ is the least number of vectors which can be added to $(\varphi_i)_{i=1}^M$ to obtain a tight frame.

Proof. A straightforward calculation shows that the sequence in (4) is indeed a λ_1-tight frame for $\mathscr{H}^N$.

For the *moreover*-part, assume that there exist vectors $(\psi_j)_{j\in J}$ with frame operator S_1 satisfying that $(\varphi_i)_{i=1}^M \cup (\psi_j)_{j\in J}$ is an A-tight frame. This implies $A \geq \lambda_1$. Now define S_2 to be the operator on $\mathscr{H}^N$ given by

$$S_2 e_j = \begin{cases} 0 : 1 \leq j \leq k, \\ (\lambda_1 - \lambda_j)e_j : k+1 \leq j \leq N. \end{cases}$$

It follows that $A \cdot Id = S + S_1$ and

$$S_1 = A \cdot Id - S \geq \lambda_1 Id - S = S_2.$$

Since S_2 has $N-k$ non-zero eigenvalues, also S_1 has at least $N-k$ non-zero eigenvalues. Hence $|J| \geq N-k$, showing that indeed $N-k$ added vectors is minimal. □

Before we delve into further explicit constructions, we need to first state some fundamental results on tight, and, in particular, Parseval frames.

The most basic invariance property a frame could have is invariance under orthogonal projections. The next result shows that this operation indeed maintains and may even improve the frame bounds. In particular, the orthogonal projection of a Parseval frame remains a Parseval frame.

Proposition 22. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with frame bounds A, B, and let P be an orthogonal projection of $\mathscr{H}^N$ onto a subspace $\mathscr{W}$. Then $(P\varphi_i)_{i=1}^M$ is a frame for $\mathscr{W}$ with frame bounds A, B.*

In particular, if $(\varphi_i)_{i=1}^M$ is a Parseval frame for $\mathscr{H}^N$ and P is an orthogonal projection on $\mathscr{H}^N$ onto $\mathscr{W}$, then $(P\varphi_i)_{i=1}^M$ is a Parseval frame for $\mathscr{W}$.

Proof. For any $x \in \mathscr{W}$,

$$A\|x\|^2 = A\|Px\|^2 \leq \sum_{i=1}^M |\langle Px, \varphi_i\rangle|^2 = \sum_{i=1}^M |\langle x, P\varphi_i\rangle|^2 \leq B\|Px\|^2 = B\|x\|^2.$$

This proves the claim. The *in particular*-part follows immediately. □

Proposition 22 immediately yields the following corollary.

Corollary 11. *Let $(e_i)_{i=1}^N$ be an orthonormal basis for $\mathscr{H}^N$, and let P be an orthogonal projection of $\mathscr{H}^N$ onto a subspace $\mathscr{W}$. Then $(Pe_i)_{i=1}^N$ is a Parseval frame for $\mathscr{W}$.*

Corollary 11 can be interpreted in the following way: Given an $M \times M$ unitary matrix, if we select any N rows from the matrix, then the column vectors from these rows form a Parseval frame for $\mathscr{H}^N$. The next theorem, known as *Naimark's Theorem*, shows that indeed every Parseval frame can be viewed as the result of this kind of operation.

Theorem 9 (Naimark's Theorem). *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with analysis operator T, let $(e_i)_{i=1}^M$ be the standard unit basis of ℓ_2^M, and let $P : \ell_2^M \to \ell_2^M$ be the orthogonal projection onto $\operatorname{ran} T$. Then the following conditions are equivalent.*

(i) $(\varphi_i)_{i=1}^M$ *is a Parseval frame for* $\mathscr{H}^N$.
(ii) For all $i = 1, \ldots, M$, *we have* $Pe_i = T\varphi_i$.
(iii) There exist $\psi_1, \ldots, \psi_M \in \mathscr{H}^{M-N}$ *such that* $(\varphi_i \oplus \psi_i)_{i=1}^M$ *is an orthonormal basis of* $\mathscr{H}^M$.

Moreover, if (iii) holds, then $(\psi_i)_{i=1}^M$ is a Parseval frame for $\mathscr{H}^{M-N}$. If $(\psi_i')_{i=1}^M$ is another Parseval frame as in (iii), then there exists a unique linear operator L on $\mathscr{H}^{M-N}$ such that $L\psi_i = \psi_i'$, $i = 1, \ldots, M$, and L is unitary.

Proof. (i)⇔(ii). By Theorem 7(iii) $(\varphi_i)_{i=1}^M$ is a Parseval frame if and only if $TT^* = P$. Therefore, (i) and (ii) are equivalent due to $T^*e_i = \varphi_i$ for all $i = 1, \ldots, M$.

(i)⇒(iii). We set $c_i := e_i - T\varphi_i$, $i = 1, \ldots, M$. Then, by (ii), $c_i \in (\operatorname{ran} T)^\perp$ for all i. Let $\Phi : (\operatorname{ran} T)^\perp \to \mathscr{H}^{M-N}$ be unitary and put $\psi_i := \Phi c_i$, $i = 1, \ldots, M$. Then, since T is isometric,

$$\langle \varphi_i \oplus \psi_i, \varphi_k \oplus \psi_k\rangle = \langle \varphi_i, \varphi_k\rangle + \langle \psi_i, \psi_k\rangle = \langle T\varphi_i, T\varphi_k\rangle + \langle c_i, c_k\rangle = \delta_{ik},$$

which proves (iii).

(iii)⇒(i). This follows directly from Corollary 11.

Concerning the *moreover*-part, it follows from Corollary 11 that $(\psi_i)_{i=1}^M$ is a Parseval frame for $\mathscr{H}^{M-N}$. Let $(\psi_i')_{i=1}^M$ be another Parseval frame as in (iii) and denote the analysis operators of $(\psi_i)_{i=1}^M$ and $(\psi_i')_{i=1}^M$ by A and A', respectively. We make use of the decomposition $\mathscr{H}^M = \mathscr{H}^N \oplus \mathscr{H}^{M-N}$. Note that both $U := (T,A)$ and $U' := (T,A')$ are unitary operators from $\mathscr{H}^M$ onto ℓ_2^M. By P_{M-N} denote the projection in $\mathscr{H}^M$ onto $\mathscr{H}^{M-N}$ and set

$$L := P_{M-N}U'^*U|_{\mathscr{H}^{M-N}} = P_{M-N}U'^*A.$$

Let $y \in \mathscr{H}^N$. Then, since $U|_{\mathscr{H}^N} = U'|_{\mathscr{H}^N} = T$, we have $P_{M-N}U'^*Uy = P_{M-N}y = 0$. Hence,

$$L\psi_i = P_{M-N}U'^*U(\varphi_i \oplus \psi_i) = P_{M-N}U'^*e_i = P_{M-N}(\varphi_i \oplus \psi_i') = \psi_i'.$$

The uniqueness of L follows from the fact that both $(\psi_i)_{i=1}^M$ and $(\psi_i')_{i=1}^M$ are spanning sets for $\mathscr{H}^{M-N}$.

To show that L is unitary, we observe that, by Proposition 10, the frame operator of $(L\psi_i)_{i=1}^M$ is given by LL^*. The claim $LL^* = Id$ now follows from the fact that also the frame operator of $(\psi_i')_{i=1}^M$ is the identity. $\square$

The simplest way to construct a frame from a given one is just to scale the frame vectors. Therefore, it seems desirable to have a characterization of the class of frames which can be scaled to a Parseval frame or a tight frame (which is equivalent). We coin such frames scalable.

Definition 21. A frame $(\varphi_i)_{i=1}^M$ for $\mathscr{H}^N$ is called (*strictly*) *scalable*, if there exist non-negative (positive, respectively) numbers $a_1,\dots,a_M$ such that $(a_i\varphi_i)_{i=1}^M$ is a Parseval frame.

The next result is closely related to Naimark's Theorem.

Theorem 10 ([117]). *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with analysis operator T. Then the following statements are equivalent.*

(i) *$(\varphi_i)_{i=1}^M$ is strictly scalable.*
(ii) *There exists a linear operator $L : \mathscr{H}^{M-N} \to \ell_2^M$ such that $TT^* + LL^*$ is a positive definite diagonal matrix.*
(iii) *There exists a sequence $(\psi_i)_{i=1}^M$ of vectors in $\mathscr{H}^{M-N}$ such that $(\varphi_i \oplus \psi_i)_{i=1}^M$ forms a complete orthogonal system in $\mathscr{H}^M$.*

We mention that Theorem 10 leads to a simple test for strict scalability in the case $M = N+1$, see [117].

If $\mathscr{H}^N$ is real then the following result applies, which can be utilized to derive a geometric interpretation of scalability. For this we once more refer to [117].

Theorem 11 ([117]). *Let $\mathscr{H}^N$ be real and let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ without zero vectors. Then the following statements are equivalent.*

(i) *$(\varphi_i)_{i=1}^M$ is not scalable.*

(ii) There exists a self-adjoint operator Y *on* $\mathscr{H}^N$ *with* $\mathrm{Tr}(Y) < 0$ *and* $\langle Y\varphi_i, \varphi_i\rangle \geq 0$ *for all* $i = 1, \ldots, M$.

(iii) There exists a self-adjoint operator Y *on* $\mathscr{H}^N$ *with* $\mathrm{Tr}(Y) = 0$ *and* $\langle Y\varphi_i, \varphi_i\rangle > 0$ *for all* $i = 1, \ldots, M$.

We finish this subsection with an existence result of tight frames with prescribed norms of the frame vectors. Its proof in [45] heavily relies on a deep understanding of the so-called frame potential and is a pure existence proof. However, in special cases constructive methods are presented in [57].

Theorem 12 ([45]). *Let* $N \leq M$*, and let* $a_1 \geq a_2 \geq \ldots \geq a_M$ *be positive real numbers. Then the following conditions are equivalent.*

(i) There exists a tight frame $(\varphi_i)_{i=1}^M$ *for* $\mathscr{H}^N$ *satisfying* $\|\varphi_i\| = a_i$ *for all* $i = 1, 2, \ldots, M$.

(ii) For all $1 \leq j < N$,

$$a_j^2 \leq \frac{\sum_{i=j+1}^M a_i^2}{N-j}.$$

(iii) We have

$$\sum_{i=1}^M a_i^2 \geq N a_1^2.$$

Equal-norm tight frames are even more desirable, but are difficult to construct. A powerful method, so-called *spectral tetris*, for such constructions was recently derived in [47], and we refer to Chapter [155]. This methodology even generates sparse frames [50], which reduce the computational complexity and also ensure high compressibility of the synthesis matrix – which then is a sparse matrix. The reader should though be cautioned that spectral tetris has the drawback that it often generates multiple copies of the same frame vector. For practical applications, this shall typically be avoided, since the frame coefficients associated with a repeated frame vector does not provide any new information about the incoming signal.

6.2 Frames with Given Frame Operator

It is often desirable to not only construct tight frames, but more generally frames with a prescribed frame operator. Typically in such a case the eigenvalues of the frame operator are given assuming that the eigenvalues are the standard unit basis. Applications are for instance noise reduction if colored noise is present.

The first comprehensive results containing necessary and sufficient conditions for the existence and the construction of tight frames with frame vectors of a prescribed norm were derived in [45] and [57], see also Theorem 12. The result in [45] was then extended in [58] to the following theorem, which now, in addition, includes prescribing the eigenvalues of the frame operator.

Theorem 13 ([58]). *Let S be a positive self-adjoint operator on $\mathscr{H}^N$, and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N > 0$ be the eigenvalues of S. Further, let $M \geq N$, and let $c_1 \geq c_2 \geq \ldots \geq c_M$ be positive real numbers. Then the following conditions are equivalent.*

(i) There exists a frame $(\varphi_i)_{i=1}^M$ for $\mathscr{H}^N$ with frame operator S satisfying $\|\varphi_i\| = c_i$ for all $i = 1, 2, \ldots, M$.
(ii) For every $1 \leq k \leq N$, we have

$$\sum_{j=1}^{k} c_j^2 \leq \sum_{j=1}^{k} \lambda_j \quad \textit{and} \quad \sum_{i=1}^{M} c_i^2 = \sum_{j=1}^{N} \lambda_j.$$

It is though often preferable to utilize equal-norm frames, since then, roughly speaking, each vector provides the same coverage for the space. In [58], it was shown that there always exists an equal-norm frame with a prescribed frame operator. This is the content of the next result.

Theorem 14 ([58]). *For every $M \geq N$ and every invertible positive self-adjoint operator S on $\mathscr{H}^N$ there exists an equal-norm frame for $\mathscr{H}^N$ with M elements and frame operator S. In particular, there exist equal norm Parseval frames with M elements in $\mathscr{H}^N$ for every $N \leq M$.*

Proof. We define the norm of the to-be-constructed frame to be c, where

$$c^2 = \frac{1}{M} \sum_{j=1}^{N} \lambda_j.$$

It is sufficient to prove that the conditions in Theorem 13(ii) are satisfied for $c_i = c$ for all $i = 1, 2, \ldots, M$. The definition of c immediately implies the second condition.

For the first condition, we observe that

$$c_1^2 = c^2 = \frac{1}{M} \sum_{j=1}^{N} \lambda_j \leq \lambda_1.$$

Hence this condition holds for $j = 1$. Now, towards a contradiction, assume that there exists some $k \in \{2, \ldots, N\}$ for which this condition fails for the first time by counting from 1 upwards, i.e.,

$$\sum_{j=1}^{k-1} c_j^2 = (k-1)c^2 \leq \sum_{j=1}^{k-1} \lambda_j, \quad \text{but} \quad \sum_{j=1}^{k} c_j^2 = kc^2 > \sum_{j=1}^{k} \lambda_j.$$

This implies

$$c^2 \geq \lambda_k \quad \text{and thus} \quad c^2 \geq \lambda_j \ \text{ for all } k+1 \leq j \leq N.$$

Hence,

$$Mc^2 \geq kc^2 + (N-k)c^2 > \sum_{j=1}^{k} \lambda_j + \sum_{j=k+1}^{N} c_j^2 \geq \sum_{j=1}^{N} \lambda_j + \sum_{j=k+1}^{N} \lambda_j = \sum_{j=1}^{N} \lambda_j,$$

which is a contradiction. The proof is completed. □

By an extension of the aforementioned algorithm *spectral tetris* [31, 48, 44, 50] to non-tight frames, Theorem 14 can be constructively realized. The interested reader is referred to Chapter [155]. We also mention that an extension of spectral tetris to construct fusion frames (cf. Section 9) exists. Further details on this topic are contained in Chapter [166].

6.3 Generic Frames

Generic frames are those optimally resilient against erasures. The precise definition is as follows.

Definition 22. A frame $(\varphi_i)_{i=1}^M$ for $\mathscr{H}^N$ is called a *generic frame*, if the erasure of any $M-N$ vectors leaves a frame, i.e., for any $I \subset \{1,\ldots,M\}$, $|I| = M-N$, the sequence $(\varphi_i)_{i=1, i\notin I}^M$ is still a frame for $\mathscr{H}^N$.

It is evident that such frames are of significant importance for applications. A first study was undertaken in [127]. Recently, using methods from algebraic geometry, equivalence classes of generic frames were extensively studied [27, 81, 136]. It was for instance shown that equivalence classes of generic frames are dense in the Grassmannian variety. For each reader to be able to appreciate these results, Chapter [157] provides an introduction to algebraic geometry followed by a survey about this and related results.

7 Frame Properties

As already discussed before, crucial properties of frames such as erasure robustness, resilience against noise, or sparse approximation properties originate from spanning and independence properties of frames [13, 14], which are typically based on the Rado-Horn Theorem [104, 129] and its redundant version [55]. These in turn are only possible because of their redundancy [12]. This section is devoted to shed light on these issues.

7.1 Spanning and Independence

As intuitively clear, the frame bounds imply certain spanning properties which are detailed in the following result. This theorem should be compared to Lemma 2, which already presented some first statements about spanning sets in frames.

Theorem 15. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ with frame bounds A and B. Then the following holds.*

(i) $\|\varphi_i\|^2 \le B_{op}$ for all $i = 1,2,\dots,M$.
(ii) If, for some $i_0 \in \{1,\dots,M\}$, we have $\|\varphi_{i_0}\|^2 = B_{op}$, then $\varphi_{i_0} \perp \mathrm{span}\{\varphi_i\}_{i=1,i\neq i_0}^M$.
(iii) If, for some $i_0 \in \{1,\dots,M\}$, we have $\|\varphi_{i_0}\|^2 < A_{op}$, then $\varphi_{i_0} \in \mathrm{span}\{\varphi_i\}_{i=1,i\neq i_0}^M$.

In particular, if $(\varphi_i)_{i=1}^M$ is a Parseval frame, then either $\varphi_{i_0} \perp \mathrm{span}\{\varphi_i\}_{i=1,i\neq i_0}^M$ (and in this case $\|\varphi_i\| = 1$) or $\|\varphi_{i_0}\| < 1$.

Proof. For any $i_0 \in \{1,\dots,M\}$ we have

$$\|\varphi_{i_0}\|^4 \le \|\varphi_{i_0}\|^4 + \sum_{i\neq i_0} |\langle \varphi_{i_0}, \varphi_i\rangle|^2 = \sum_{i=1}^M |\langle \varphi_{i_0}, \varphi_i\rangle|^2 \le B_{op}\|\varphi_{i_0}\|^2. \tag{5}$$

The claims (i) and (ii) now directly follow from (5).

(iii). Let P denote the orthogonal projection of $\mathscr{H}^N$ onto $(\mathrm{span}\{\varphi_i\}_{i=1,i\neq i_0}^M)^\perp$. Then

$$A_{op}\|P\varphi_{i_0}\|^2 \le \|P\varphi_{i_0}\|^4 + \sum_{i=1,i\neq i_0}^M |\langle P\varphi_{i_0}, \varphi_i\rangle|^2 = \|P\varphi_{i_0}\|^4.$$

Hence, either $P\varphi_{i_0} = 0$ (and thus $\varphi_{i_0} \in \mathrm{span}\{\varphi_i\}_{i=1,i\neq i_0}^M$) or $A_{op} \le \|P\varphi_{i_0}\|^2 \le \|\varphi_{i_0}\|^2$. This proves (iii). □

Ideally, we are interested in having an exact description of a frame in terms of its spanning and independence properties. One could think of the following questions to be answered by such a measure: How many disjoint linearly independent spanning sets does the frame contain? After removing these, how many disjoint linearly independent sets which span hyperplanes does it contain? And many more.

One of the main results in this direction is the following from [14].

Theorem 16. *Every unit-norm tight frame $(\varphi_i)_{i=1}^M$ for $\mathscr{H}^N$ with $M = kN + j$ elements, $0 \le j < N$, can be partitioned into k linearly independent spanning sets plus a linearly independent set of j elements.*

For its proof and further related results we refer to Chapter [156].

7.2 Redundancy

As we have discussed and will be seen throughout this book, redundancy is the key property of frames. This fact makes it even more surprising that until recently not much attention has been paid to introduce meaningful quantitative measures of redundancy. The classical measure of the *redundancy* of a frame $(\varphi_i)_{i=1}^M$ for $\mathscr{H}^N$ is the quotient of the number of frame vectors and the dimension of the ambient space, i.e., $\frac{M}{N}$. This measure has however serious problems to distinguish, for instance, the two frames in Example 2 (1) and (2) by assigning the same redundancy measure $\frac{2N}{N} = 2$ to both of those. From a frame perspective these two frames are very different, since, for instance, one contains two spanning sets whereas the other just contains one.

Recently, in [12] a new notion of redundancy was proposed which seems to better capture the spirit of what redundancy should represent. To present this notion, let $\mathbb{S} = \{x \in \mathscr{H}^N : \|x\| = 1\}$ denote the unit sphere in $\mathscr{H}^N$, and let $P_{\mathrm{span}\{x\}}$ denote the orthogonal projection onto the subspace span$\{x\}$ for some $x \in \mathscr{H}^N$.

Definition 23. Let $\Phi = (\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$. For each $x \in \mathbb{S}$, the *redundancy function* $\mathscr{R}_\Phi : \mathbb{S} \to \mathbb{R}^+$ is defined by

$$\mathscr{R}_\Phi(x) = \sum_{i=1}^M \|P_{\mathrm{span}\{\varphi_i\}} x\|^2.$$

Then the *upper redundancy* of Φ is defined by

$$\mathscr{R}_\Phi^+ = \max_{x \in \mathbb{S}} \mathscr{R}_\Phi(x),$$

and the *lower redundancy* of Φ is defined by

$$\mathscr{R}_\Phi^- = \min_{x \in \mathbb{S}} \mathbb{R}_\Phi(x).$$

Moreover, Φ has *uniform redundancy*, if

$$\mathscr{R}_\Phi^- = \mathscr{R}_\Phi^+.$$

One might hope that this new notion of redundancy provides information about spanning and independence properties of the frame, since these are closely related to questions such as whether a frame is resilient with respect to deletion of a particular number of frame vectors, say. And, indeed, such a link exists and is detailed in the next result.

Theorem 17 ([12]). *Let $\Phi = (\varphi_i)_{i=1}^M$ be a frame for $\mathscr{H}^N$ without zero vectors. Then the following conditions hold.*

(i) Φ contains $\lfloor \mathscr{R}_\Phi^- \rfloor$ disjoint spanning sets.
(ii) Φ can be partitioned into $\lceil \mathscr{R}_\Phi^+ \rceil$ linearly independent sets.

Various other properties of this notion of redundancy are known such as additivity or its range, and we refer to [12] and Chapter [156] for more details.

At this point, we would just like to point out that another interpretation of upper and lower redundancy is possible, since it coincides with the optimal frame bounds of the normalized frame $(\frac{\varphi}{\|\varphi\|})_{i=1}^{M}$ – after deletion of zero vectors. The crucial point to make is that with this viewpoint Theorem 17 combines analytic and algebraic properties of Φ.

7.3 Equivalence of Frames

We now consider equivalence classes of frames. As in other research areas, the idea being that frames in the same equivalence class share certain properties.

7.3.1 Isomorphic Frames

The following definition states one equivalence relation for frames.

Definition 24. Two frames $(\varphi_i)_{i=1}^{M}$ and $(\psi_i)_{i=1}^{M}$ for $\mathscr{H}^N$ are called *isomorphic*, if there exists an operator $F : \mathscr{H}^N \to \mathscr{H}^N$ satisfying $F\varphi_i = \psi_i$ for all $i = 1, 2, \ldots, M$.

We remark that – due to the spanning property of frames – an operator F as in the above definition is both invertible and unique. Moreover, we mention that in [4] the isomorphy of frames with an operator F as above was termed F-equivalence.

The next theorem characterizes the isomorphy of two frames in terms of their analysis and synthesis operators.

Theorem 18. *Let $(\varphi_i)_{i=1}^{M}$ and $(\psi_i)_{i=1}^{M}$ be frames for $\mathscr{H}^N$ with analysis operators T_1 and T_2, respectively. Then the following conditions are equivalent.*

(i) $(\varphi_i)_{i=1}^{M}$ *is isomorphic to* $(\psi_i)_{i=1}^{M}$.
(ii) $\operatorname{ran} T_1 = \operatorname{ran} T_2$.
(iii) $\ker T_1^* = \ker T_2^*$.

If one of (i)–(iii) holds then the operator $F : \mathscr{H}^N \to \mathscr{H}^N$ with $F\varphi_i = \psi_i$ for all $i = 1, \ldots, N$ is given by $F = T_2^*(T_1^*|_{\operatorname{ran} T_1})^{-1}$.

Proof. The equivalence of (ii) and (iii) follows by orthogonal complementation. In the following let $(e_i)_{i=1}^{M}$ denote the standard unit vector basis of ℓ_2^M.

(i)⇒(iii). Let F be an invertible operator on $\mathscr{H}^N$ such that $F\varphi_i = \psi_i$ for all $i = 1, \ldots, M$. Then Proposition 9 implies $T_2 = T_1 F^*$ and hence $F T_1^* = T_2^*$. Since F is invertible, (iii) follows.

(ii)⇒(i). Let P be the orthogonal projection onto $\mathscr{W} := \operatorname{ran} T_1 = \operatorname{ran} T_2$. Then $\varphi_i = T_1^* e_i = T_1^* P e_i$ and $\psi_i = T_2^* e_i = T_2^* P e_i$. The operators T_1^* and T_2^* both map $\mathscr{W}$ bijectively onto $\mathscr{H}^N$. Therefore, the operator $F := T_2^*(T_1^*|_{\mathscr{W}})^{-1}$ maps $\mathscr{H}^N$ bijectively onto itself. Consequently, for each $i \in \{1, \ldots, M\}$ we have

$$F\varphi_i = T_2^*(T_1^*|_{\mathscr{W}})^{-1} T_1^* P e_i = T_2^* P e_i = \psi_i,$$

which proves (i) as well as the additional statement on the operator F. □

An obvious, though interesting result in the context of frame isomorphy is that the Parseval frame in Lemma 7 is in fact isomorphic to the original frame.

Lemma 8. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathcal{H}^N$ with frame operator S. Then the Parseval frame $(S^{-1/2}\varphi_i)_{i=1}^M$ is isomorphic to $(\varphi_i)_{i=1}^M$.*

Similarly, a given frame is also isomorphic to its canonical dual frame.

Lemma 9. *Let $(\varphi_i)_{i=1}^M$ be a frame for $\mathcal{H}^N$ with frame operator S. Then the canonical dual frame $(S^{-1}\varphi_i)_{i=1}^M$ is isomorphic to $(\varphi_i)_{i=1}^M$.*

Intriguingly, it turns out – and will be proven in the following result – that the canonical dual frame is the only dual frame which is isomorphic to a given frame.

Proposition 23. *Let $\Phi = (\varphi_i)_{i=1}^M$ be a frame for $\mathcal{H}^N$ with frame operator S, and let $(\psi_i)_{i=1}^M$ and $(\tilde{\psi}_i)_{i=1}^M$ be two different dual frames for Φ. Then $(\psi_i)_{i=1}^M$ and $(\tilde{\psi}_i)_{i=1}^M$ are not isomorphic.*

In particular, $(S^{-1}\varphi_i)_{i=1}^M$ is the only dual frame for Φ which is isomorphic to Φ.

Proof. Let $(\psi_i)_{i=1}^M$ and $(\tilde{\psi}_i)_{i=1}^M$ be different dual frames for Φ. Towards a contradiction, we assume that $(\psi_i)_{i=1}^M$ and $(\tilde{\psi}_i)_{i=1}^M$ are isomorphic, and let F denote the invertible operator satisfying $\psi_i = F\tilde{\psi}_i$, $i = 1, 2, \ldots, M$. Then, for each $x \in \mathcal{H}^N$ we have

$$F^*x = \sum_{i=1}^M \langle F^*x, \tilde{\psi}_i\rangle \varphi_i = \sum_{i=1}^M \langle x, F\tilde{\psi}_i\rangle \varphi_i = \sum_{i=1}^M \langle x, \psi_i\rangle \varphi_i = x.$$

Thus, $F^* = Id$ which implies $F = Id$, a contradiction. □

7.3.2 Unitarily Isomorphic Frames

A stronger version of equivalence is given by the notion of unitarily isomorphic frames.

Definition 25. Two frames $(\varphi_i)_{i=1}^M$ and $(\psi_i)_{i=1}^M$ for $\mathcal{H}^N$ are *unitarily isomorphic*, if there exists a unitary operator $F : \mathcal{H}^N \to \mathcal{H}^N$ satisfying $F\varphi_i = \psi_i$ for all $i = 1, 2, \ldots, M$.

In the situation of Parseval frames though, the notions of isomorphy and unitary isomorphy coincide.

Lemma 10. *Let $(\varphi_i)_{i=1}^M$ and $(\psi_i)_{i=1}^M$ be isomorphic Parseval frames for $\mathcal{H}^N$. Then they are even unitarily isomorphic.*

Proof. Let F be an invertible operator on $\mathcal{H}^N$ with $F\varphi_i = \psi_i$ for all $i = 1, 2, \ldots, M$. By Proposition 10, the frame operator of $(F\varphi_i)_{i=1}^M$ is $FIdF^* = FF^*$. On the other hand, the frame operator of $(\psi_i)_{i=1}^M$ is the identity. Hence, $FF^* = Id$. □

We end this section with a necessary and sufficient condition for two frames to be unitarily isomorphic.

Proposition 24. *For two frames $(\varphi_i)_{i=1}^M$ and $(\psi_i)_{i=1}^M$ for $\mathscr{H}^N$ with analysis operators T_1 and T_2, respectively, the following conditions are equivalent.*

(i) $(\varphi_i)_{i=1}^M$ *and* $(\psi_i)_{i=1}^M$ *are unitarily isomorphic.*
(ii) $\|T_1^* c\| = \|T_2^* c\|$ *for all* $c \in \ell_2^M$.
(iii) $T_1 T_1^* = T_2 T_2^*$.

Proof. (i)⇒(iii). Let F be a unitary operator on $\mathscr{H}^N$ with $F\varphi_i = \psi_i$ for all $i = 1,\ldots,M$. Then, since by Proposition 9 we have $T_2 = T_1 F^*$, we obtain $T_2 T_2^* = T_1 F^* F T_1^* = T_1 T_1^*$ and thus (iii).

(iii)⇒(ii). This is immediate.

(ii)⇒(i). Since (ii) implies $\ker T_1^* = \ker T_2^*$, it follows from Theorem 18 that $F\varphi_i = \psi_i$ for all $i = 1,\ldots,M$, where $F = T_2^*(T_1^*|_{\operatorname{ran} T_1})^{-1}$. But this operator is unitary since (ii) also implies

$$\|T_2^*(T_1^*|_{\operatorname{ran} T_1})^{-1}x\| = \|T_1^*(T_1^*|_{\operatorname{ran} T_1})^{-1}x\| = \|x\|$$

for all $x \in \mathscr{H}^N$. □

8 Applications of Finite Frames

Finite frames are a versatile methodology for any application which requires redundant, yet stable decompositions. For instance, for analysis or transmission of signals, but surprisingly also for more theoretically oriented questions. We state some such applications in this section, which also coincide with the chapters of this book.

8.1 Noise and Erasure Reduction

Noise and erasures are one of the most common problems signal transmissions have to face [131, 132, 133]. The redundancy of frames is particularly suitable to reduce and compensate for such disturbances. Pioneering studies can be found in [51, 94, 95, 96], followed by the fundamental papers [10, 16, 103, 137, 150]. In addition one is always faced with the problem of suppressing errors introduced through quantization, both PCM [21, 152] and Sigma-Delta quantization [7, 8, 17, 18]. Theoretical error considerations range from worst to average case scenarios. Different strategies for reconstruction exist depending on whether the receiver is aware or unaware of noise and erasures. Some more recent work also takes special types of erasures [19] or the selection of dual frames for reconstruction [124, 122] into account. Chapter [160] provides a comprehensive survey on these considerations and related results.

8.2 Resilience against Perturbations

Perturbations of a signal are an additional problem faced by signal processing applications. Various results on the ability of frames to be resilient against perturbations are known. One class focusses on generally applicable frame perturbations results [3, 69, 38, 60], some even in the Banach space setting [40, 69]. Yet another topic are perturbations of specific frames such as Gabor frames [41], frames containing a Riesz basis [39], or frames for shift-invariant spaces [154]. Finally, also extensions such as fusion frames are studied with respect to their behavior under perturbations [53].

8.3 Quantization Robustness

Each signal processing application contains an analog-to-digital conversion step, which is called quantization. Quantization is typically applied to the transform coefficients, which in our case are (redundant) frame coefficients, see [95, 96]. Interestingly, the redundancy of the frame can be successfully explored in the quantization step by using so-called Sigma-Delta algorithms and a particular non-canonical dual frame reconstruction. In most regimes, the performance is significantly better than rounding each coefficient separately (PCM). This was first observed in [7, 8]. Within a short amount of time, the error bounds were improved [17, 115], refined quantization schemes were studied [18, 15], specific dual frame constructions for reconstruction were developed [9, 99, 119], and also PCM was revisited [106, 152]. The interested reader is referred to Chapter [161], which provides an introduction to quantization of finite frames.

8.4 Compressed Sensing

Since high dimensional signals are typically concentrated on lower dimensional subspaces, it is a natural assumption that the collected data can be represented by a sparse linear combination of an appropriately chosen frame. The novel methodology of Compressed Sensing, initially developed in [33, 34, 79], utilizes this observation to show that such signals can be reconstructed from very few non-adaptive linear measurements by linear programming techniques. For an introduction, we refer to the books [85, 87] and the survey [26]. Finite frames thus play an essential role, both as sparsifying systems and in designing the measurement matrix. For a selection of studies focussing in particular on the connection to frames, we refer to [1, 2, 32, 70, 142, 143], and on the connection to structured frames such as fusion frames, see [23, 86]. Chapter [162] provides an introduction into Compressed Sensing and the connection to finite frame theory.

We wish to mention that there exists yet another intriguing connection of finite frames to sparsity methodologies, namely, aiming for sparse frame vectors to ensure low computational complexity. For this, we refer to the two papers [31, 50] and to Chapter [166].

8.5 Filter Banks

Filter banks are the basis for most signal processing applications. We exemplarily mention the general books [126, 146] and those with a particular focus on wavelets [76, 135, 151], as well as the beautiful survey articles [110, 111]. Usually, several filters are applied in parallel to an input signal, followed by downsampling. This processing method is closely related to finite frame decomposition provided that the frame consists of equally-spaced translates of a fixed set of vectors, first observed in [20, 22, 72, 73] and later refined and extended in [63, 64, 91, 113]. This viewpoint has the benefit of providing a deeper understanding of filtering procedures, while containing the potential of extensions of classical filter bank theory. We refer to Chapter [163] which provides an introduction into filter banks and their connections with finite frame theory.

8.6 Stable Partitions

The Feichtinger conjecture in frame theory conjectures the existence of certain partitions of frames into sequences with "good" frame bounds, see [42]. Its relevance becomes evident when modeling distributed processing and stable frames are required for the local processing units (see also Section 9 on fusion frames). The fundamental papers [62, 49, 56] then linked this conjecture to a variety of open conjectures in what is customarily coined pure mathematics such as the Kadison-Singer Problem in C^*-Algebras [108]. Chapter [164] provides an introduction into these connections and their significance. It should be mentioned that a particular focus of this chapter is also on the so-called Paulsen problem [11, 28, 46], which provides error estimates on the ability of a frame to be simultaneously (almost) equal-norm and (almost) tight.

9 Extensions

Typically motivated by applications, various extensions of finite frame theory have been developed over the last years. In this book, the chapters [165] and [166] are devoted to the main two generalizations, whose key ideas we will now briefly describe.

- *Probabilistic Frames.* This theory is based on the observation that finite frames can be regarded as mass points distributed in $\mathscr{H}^N$. As an extension, probabilistic frames, which were introduced and studied in [82, 83, 84], constitute a class of general probability measures again with appropriate stability constraints. Applications include, for instance, directional statistics in which probabilistic frames can be utilized to measure inconsistencies of certain statistical tests [109, 144, 145]. For more details on the theory and applications of probabilistic frames, we refer to Chapter [165].
- *Fusion Frames.* Signal processing by finite frames can be regarded as projections onto one-dimensional subspaces. In contrast to this, fusion frames, which were introduced in [52, 54], to analyze and process a signal by (orthogonal) projections onto multi-dimensional subspaces, which again have to satisfy some stability conditions. They also allow for a local processing in the different subspaces. This theory is in fact a perfect fit to applications requiring distributed processing, and we refer to the series of papers [23, 24, 29, 31, 43, 47, 44, 64, 118, 125]. We should also mention that a closely related generalization called G-frames exists, which however does not admit any additional (local) structure and which is unrelated to applications (see, for instance, [138, 139]). A detailed introduction into fusion frame theory can be found in Chapter [166].

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