AMBIGUITY FUNCTIONS, WIGNER DISTRIBUTIONS AND COHEN'S CLASS FOR LCA GROUPS

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Abstract. In this paper we construct a general class of time-frequency representations for LCA groups which parallel Cohen's class for the real line. For this, we generalize the notion of ambiguity function and Wigner distribution to the setting of general LCA groups in such a way that the Plancherel transform of the ambiguity function coincides with the Wigner distribution. Furthermore, properties of the general ambiguity function and Wigner distribution are studied. In detail we characterize those groups whose ambiguity functions and Wigner distributions vanish at infinity or are square-integrable. Finally, we explicitely construct Cohen's class for the group of p-adic numbers, p prime.

1. Introduction

For $f, g \in L^2(\mathbb{R})$, the *Wigner distribution* is the function $W_{f,g}$ on $\mathbb{R} \times \mathbb{R}$ defined by

$$W_{f,g}(y,x) = \int_{\mathbb{R}} \overline{f\left(x - \frac{t}{2}\right)} g\left(x + \frac{t}{2}\right) e^{-2\pi i y t}\, dt.$$

In 1932, it was introduced by Wigner [16] in connection with quantum mechanics. The concept was reintroduced by Ville [15] in signal analysis some 15 years later. The Wigner distribution became a popular tool in engineering through the influential work of Claasen and Mecklenbräuker [3, 4, 5]. Since the Wigner distribution is not always positive, which is desirable for a time-frequency representation, other time-frequency representations were constructed.

In 1966, Cohen [6] introduced a general class of time-frequency representations that allow us to pick time-frequency representations with prescribed, desirable properties. Cohen's class is defined by using the so-called "kernel method" [7, Chapter 9]. This method makes extensive usage of the fact that the Wigner distribution coincides with the Plancherel transform of the ambiguity function.

In the early 1950s, Woodward introduced the ambiguity function on $\mathbb{R}$ for radar analysis [18]. The *ambiguity function* is the function $A_{f,g}$ defined on $\mathbb{R} \times \mathbb{R}$ by

$$A_{f,g}(x,y) = \int_{\mathbb{R}} \overline{f\left(t - \frac{x}{2}\right)} g\left(t + \frac{x}{2}\right) e^{2\pi i y t}\, dt,$$

where $f, g \in L^2(\mathbb{R})$. Since the fundamental work of Wilcox [17], the ambiguity function has been widely used in the context of radar and sonar (compare [14]).

1991 *Mathematics Subject Classification.* Primary 94A12, 43A32; Secondary 43A15, 43A40.

Key words and phrases. Ambiguity function, Wigner distribution, Cohen's class, locally compact abelian group.

In the last 20 years there have been numerous attempts to define both time-frequency representations on groups other than $\mathbb{R}$, for example, on $\mathbb{Z}$, on finite abelian groups and, more generally, on elementary LCA groups [1, 4, 8, 13]. But until now there has been no attempt to define Cohen's class for other settings than the real line.

In the present paper we construct a general class of time-frequency representations for LCA groups which parallel Cohen's class for the real line by generalizing the notion of ambiguity function and Wigner distribution to LCA groups (locally compact abelian groups, which we always assume to be second countable) in an appropriate way. The main problem is the existence of LCA groups for which the mapping $x \mapsto 2x$ is no longer an automorphism. We overcome this difficulty by using the concept of "2-root-compactness" as a substitude in those cases.

In Subsection 3.1 we give the definition of ambiguity function and Wigner distribution in the general setting and examine their elementary properties. It turns out that all basic properties which hold in the real case remain true in the general situation. Furthermore, in Subsection 3.2 and 3.3 we establish results concerning the behaviour at infinity and the square-integrability. In the last part of Section 3 we then prove that the Plancherel transform of the ambiguity function coincides with the Wigner distribution for a large class of LCA groups.

The purpose of the fourth section is to construct a class of time-frequency representations for LCA groups which parallel Cohen's class for $G = \mathbb{R}$ by using the results from the previous section.

In the last section we discuss the example of the group of p-adic numbers, where p is a prime. These groups seem to be the right setting for problems of computer science, because the group laws imitate the computer arithmetic most closely. We study properties of their ambiguity functions and Wigner distributions and give a concrete formula for Cohen's class.

2. Preliminaries and Notation

Now let G be a locally compact abelian (LCA) group with its dual group denoted by $\widehat{G}$. We always assume G to be second countable. The neutral elements of G and $\widehat{G}$ are denoted by e and 1, respectively. For a function f on G, one defines $L_y f(x) = f(y^{-1}x)$ and $f^*(x) = \overline{f(x^{-1})}$ for all $x, y \in G$. Let f and g be measurable functions on G. Then the convolution product $f * g$ of f and g is defined by

$$f * g(x) = \int_G f(y) g(y^{-1}x)\, dy,$$

whenever this makes sense. For $M \subseteq G$, the characteristic function of M is denoted by χ_M. Let S be some set. Then Id_S denotes the identity operator.

Let G_0 denote the connected component of the identity in G. We call G a *Lie group*, if G_0 is an open subgroup of G which is topologically isomorphic to a group of the form $\mathbb{R}^p \times \mathbb{T}^r$, $p, r \geq 0$.

G is called *n-root compact* for some $n \in \mathbb{N}$ if, for each compact subset C of G, the set

$$\{x \in G : x^n \in C\}$$

is compact. It should be mentioned that this definition is equivalent to [12, Definition 3.1.1] by [12, Theorem 3.1.4]. Note that every compactly generated LCA group is n-root compact for each $n \in \mathbb{N}$ [12, Example 3.1.3].

In the following $G^{(2)}$ is the subgroup of G defined by $G^{(2)} = \{x^2 : x \in G\}$. Let the Fourier transformation $\widehat{\ } : L^1(G) \to C_0(\widehat{G})$, $f \mapsto \widehat{f}$ be defined by

$$\widehat{f}(\omega) = \int_G f(t)\overline{\omega(t)}\, dt.$$

When Haar measures on G and $\widehat{G}$ are suitably normalized, the Fourier transformation on $L^1(G) \cap L^2(G)$ extends to a unitary operator from $L^2(G)$ to $L^2(\widehat{G})$, the so-called *Plancherel transformation*. We also denote this transformation by $\widehat{\ }$.

As an extensive reference to duality theory of general LCA groups we mention [10].

The following lemma will be needed subsequently several times.

Lemma 2.1. *Let G be a LCA group and H an open subgroup such that there exists a topological isomorphism $\Phi : G \to H$.*

(i) *There exists a positive constant c such that*

$$\int_G |f(\Phi(t))|\, dt \le c \int_G |f(t)|\, dt \quad \textit{for all } f \in L^1(G).$$

(ii) *Let the Haar measure on H be induced by the Haar measure on G. Then there exists a positive constant d such that*

$$\int_G f(t)\, dt = d \int_H f(\Phi^{-1}(t))\, dt \quad \textit{for all } f \in L^1(G).$$

Proof. Let I_G denote the Haar integral on G. We consider the linear functional

$$J : C_c(H) \to \mathbb{C}, \quad J(f) := I_G(f \circ \Phi).$$

Since this functional is translation-invariant and Haar integrals are unique up to a positive multiplicative constant, we obtain (i).

The claim in (ii) follows immediately from the fact that $\Phi : G \to H$ is a topological isomorphism. □

3. The general ambiguity function and Wigner distribution

Our aim is to generalize Cohen's class to the setting of LCA groups. This requires a definition of ambiguity functions $A_{f,g}$ and Wigner distributions $W_{f,g}$ in this general situation such that the relation $\widehat{A_{f,g}} = W_{f,g}$ is satisfied. There already exist definitions of ambiguity functions in more general settings, for example, in [8, Subsection 7.6.1] for elementary LCA groups or in [1, Section 10.2] for finite abelian groups. But defining the Wigner distribution to be the Plancherel transform of this ambiguity function yields a distribution which is no generalization of the classical Wigner distribution.

3.1. **Definition and some basic facts.** We start by giving the definition of ambiguity functions and Wigner distributions in the general setting, discussing the hypotheses needed and stating some elementary properties.

Definition 3.1. Let G be a LCA group, let H be an open subgroup such that there exists a topological isomorphism $\Phi : G \to H$ and let $f, g \in L^2(G)$. The *ambiguity function associated with* H *and* Φ *of* f *and* g on $G \times \widehat{G}$ is defined by

$$A_{f,g}(x,\omega) = \int_G \overline{f(t\Phi(x^{-1}))}g(t\Phi(x))\omega(t)\,dt \quad ((x,\omega) \in G \times \widehat{G}).$$

The *Wigner distribution associated with* H *and* Φ *of* f *and* g on $\widehat{G} \times G$ is defined by

$$W_{f,g}(\omega,x) = \int_G \overline{f(x\Phi(t^{-1}))}g(x\Phi(t))\overline{\omega(t)}\,dt \quad ((\omega,x) \in \widehat{G} \times G).$$

Further, we denote $A_{f,f}$ by A_f and $W_{f,f}$ by W_f.

It it easily checked, by using Hölder's inequality and Lemma 2.1 (i), that $A_{f,g}(x,\omega)$ and $W_{f,g}(\omega,x)$ are defined for each $x \in G$, $\omega \in \widehat{G}$.

Remark 3.2. In Subsection 3.2 and 3.3 we will see that the choice of H and Φ does not affect whether the ambiguity function or Wigner distribution vanishes at infinity or is square-integrable. But it affects the properties of the members of Cohen's class (compare Section 4).

In the following we will state all possible choices of H and Φ for several LCA groups.

Example 3.3. (i) Consider the case $G = \mathbb{R}$. By taking $H = \mathbb{R}$ and defining $\Phi : \mathbb{R} \to \mathbb{R}$ by $\Phi(x) = \frac{x}{2}$, we see that the definitions above generalize the definition of the classical ambiguity function and Wigner distribution on $L^2(\mathbb{R})$.

(ii) For $G = \mathbb{R}^p$ or $\mathbb{T}^r$, an open subgroup H of G always equals G. So all possible topological isomorphisms Φ may be found in [10, Example 26.18 (h) and (i)].

(iii) For $G = \mathbb{Z}^q$, H is an arbitrary subgroup of G. It is easy to check (see also [10, Example 26.18 (g)]) that each topological isomorphism $\Phi : G \to H$ is given by an element of the discrete group of $q \times q$ matrices A having integer entries and for which $\det A \neq 0$.

(iv) For G finite, H always equals G. By [10, Example 23.27 (d)], G is isomorphic to $\mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_s}$ for integers $m_1, \ldots, m_s$ greater than 1, each of which is a power of a prime. Now let $G = \mathbb{Z}_m$, $m > 1$ and a power of a prime, and let $a, b \in G$ be generators of G. Then $\Phi : G \to G$, $\Phi(a^n) = b^n$, $n \in \mathbb{Z}$, is a topological isomorphism and each of the topological automorphisms of G can be constructed in such a way.

Next we will discuss why H has to be open and when it is open automatically.

Remark 3.4. (i) We need the openness of H because of the following reason. Suppose that H is not open. Then the set $H = \{\Phi(t) : t \in G\}$ is a zero set in G. Let $f, g \in L^2(G)$ be such that $f(t) = g(t^{-1})$ for all $t \in H$. Then the Wigner distribution of f and g may not be defined at $(1, e)$, since

$$W_{f,g}(1,e) = \int_G \overline{f(\Phi(t^{-1}))} g(\Phi(t))\, dt = \int_G |f(\Phi(t))|^2\, dt.$$

(ii) If G is an arbitrary LCA group, H a closed subgroup of G and $\Phi : G \to H$ a topological isomorphism, we cannot always conclude that H is open. A counterexample is the group $G := \prod_{i=1}^{\infty} \mathbb{Z}_2$, where $\mathbb{Z}_2$ denotes the additive group of residues mod 2, endowed with the product topology. Here we choose H to be

$$H = \prod_{i=1}^{\infty} F_i, \quad \text{where } F_i = \begin{cases} \{0\} & : \ i \text{ is odd}, \\ \mathbb{Z}_2 & : \ i \text{ is even} \end{cases}$$

and define $\Phi : G \to H$ by

$$\Phi((x_n)_{n\in\mathbb{N}}) = (y_n)_{n\in\mathbb{N}}, \quad \text{where } y_n = \begin{cases} 0 & : \ n \text{ is odd}, \\ x_{\frac{n}{2}} & : \ n \text{ is even}. \end{cases}$$

Obviously, Φ is a topological isomorphism, but H is not open in G.

(iii) Let G be a LCA Lie group, H a closed subgroup of G and $\Phi : G \to H$ a topological isomorphism. It is well-known that this implies H to be open. Hence in this case the subgroup H is open automatically.

For the remainder of this paper let H be an open subgroup of G such that there exists a topological isomorphism $\Phi : G \to H$.

Next we state some important basic properties of the ambiguity function and Wigner distribution which parallel the known properties on $\mathbb{R}$. The proofs carry over in a straightforward manner.

Proposition 3.5. *Let G be a LCA group. For all $f, g \in L^2(G)$ and $x \in G$, $\omega \in \widehat{G}$, the following hold.*

(i) $|A_{f,g}(x,\omega)| \le \|f\|_2 \|g\|_2$ *and* $|W_{f,g}(\omega,x)| \le c\|f\|_2\|g\|_2$ *for some* $c > 0$.
(ii) $\overline{A_{f,g}(x,\omega)} = A_{g,f}(x^{-1}, \overline{\omega})$ *and* $\overline{W_{f,g}(\omega,x)} = W_{g,f}(\omega,x)$.
(iii) $A_f(e,1) \ge 0$.

To conclude this subsection we want to investigate the ambiguity function and the Wigner distribution with respect to continuity. It will turn out that we obtain analogous results as for the classical ambiguity function and Wigner distribution.

Proposition 3.6. *Let G be a LCA group.*

(i) *For all $f, g \in L^2(G)$,*

$$A_{f,g} \in C(G \times \widehat{G}) \quad \text{and} \quad W_{f,g} \in C(\widehat{G} \times G).$$

(ii) *The mappings*

$$(f,g) \mapsto A_{f,g}, \quad L^2(G) \times L^2(G) \to (C(G \times \widehat{G}), \|\cdot\|_\infty)$$

$$\text{and} \quad (f,g) \mapsto W_{f,g}, \quad L^2(G) \times L^2(G) \to (C(\widehat{G} \times G), \|\cdot\|_\infty)$$

are continuous.

Proof. Using Proposition 3.5 (i) and the continuity of Φ, the claim concerning the ambiguity functions follows immediately. To prove the analogous result for the Wigner distribution, we have to use Lemma 2.1 (ii) in addition. □

3.2. **Behaviour at infinity.** In what follows, we are interested in the behaviour at infinity of the ambiguity function and the Wigner distribution. It will turn out, that we can classify exactly those LCA groups, for which the ambiguity function or the Wigner distribution is continuous and vanishes at infinity.

Theorem 3.7. *Let G be a LCA group. Then the following conditions are equivalent.*

(i) $A_{f,g} \in C_0(G \times \widehat{G})$ *for all* $f, g \in L^2(G)$.

(ii) $W_{f,g} \in C_0(\widehat{G} \times G)$ *for all* $f, g \in L^2(G)$.

(iii) G *is 2-root compact.*

Proof. We start by proving (i) $\Leftrightarrow$ (iii). Suppose that (i) holds. We argue by contradiction and assume that G is not 2-root compact. This implies the existence of a compact subset K of G such that the set $X := \{x \in G : x^2 \in K\}$ is non-compact. Let V be any compact neighbourhood of e and define $f, g \in L^2(G)$ by $f = \chi_V$ and $g = \chi_{V\Phi(K)}$. Then, for all $x \in X$, we obtain

$$|A_{f,g}(x,1)| = \left|\int_G \overline{f(t)} g(t\,\Phi(x^2))\,dt\right| = \left|\int_G \chi_V(t)\chi_{V\Phi(K)}(t\,\Phi(x^2))\,dt\right|$$

$$= \left|\int_G \chi_{V \cap V\Phi(x^{-2}K)}(t)\,dt\right| = |V|,$$

a contradiction.

Now suppose that G is 2-root compact. Let $f, g \in L^2(G)$. Proposition 3.6 (i) shows that $A_{f,g} \in C(G \times \widehat{G})$. Let $\epsilon > 0$. For all $(x,\omega) \in G \times \widehat{G}$, we obtain

$$|A_{f,g}(x,\omega)| \le \int_G |f(t)||g(t\,\Phi(x^2))|\,dt = (|f^*| * |g|)(\Phi(x^2)).$$

Now $f, g \in L^2(G)$ implies $|f^*| * |g| \in C_0(G)$. Hence there exists a compact set $\tilde{K} \subseteq G$ such that

$$(|f^*| * |g|)(y) < \frac{\epsilon}{2} \quad \text{for all } y \in G \backslash \tilde{K}.$$

Define $K \subseteq G$ by $K = \{x \in G : x^2 \in \Phi^{-1}(\tilde{K})\}$. Then

$$|A_{f,g}(x,\omega)| \le (|f^*| * |g|)(\Phi(x^2)) < \frac{\epsilon}{2} \quad \text{for all } x \in G\backslash K,\ \omega \in \widehat{G}.$$

Since G is 2-root compact and Φ is a topological isomorphism, K is compact.

On the other hand, $A_{f,g}$ may be rewritten as

$$A_{f,g}(x,\omega) = ((L_{\Phi(x)}\overline{f}) \cdot (L_{\Phi(x^{-1})} g))\widehat{\ }(\overline{\omega}) \quad ((x,\omega) \in G \times \widehat{G}).$$

Since both functions $L_{\Phi(x)}\overline{f}$ and $L_{\Phi(x^{-1})}g$ are square-integrable, we obtain $(L_{\Phi(x)}\overline{f})\cdot(L_{\Phi(x^{-1})}g) \in L^1(G)$. Hence $A_{f,g}(x,\cdot) \in C_0(\widehat{G})$ for each $x \in G$. This implies that, for each $x \in G$, there exists a compact subset $\Gamma(x) \subseteq \widehat{G}$ such that

$$|A_{f,g}(x,\omega)| < \frac{\epsilon}{2} \quad \text{for all } \omega \in \widehat{G}\backslash\Gamma(x).$$

Now we use the sets $\Gamma(x)$, $x \in G$, to construct a compact subset Γ of $\widehat{G}$ such that $|A_{f,g}(x,\omega)| < \epsilon$ for all $(x,\omega) \in (G \times \widehat{G})\backslash(K \times \Gamma)$.

For this, let $(x,\omega) \in G \times \widehat{G}$ and let x_0 be an arbitrarily fixed element of G. We obtain

$$\begin{aligned} |A_{f,g}(x,\omega)| &= |\langle \omega \cdot L_{\Phi(x^{-2}x_0^2)\Phi(x_0^{-2})}g, f\rangle| \\ &= |A_{f,g}(x_0,\omega) + \langle \omega \cdot L_{\Phi(x_0^{-2})}g, L_{\Phi(x^2x_0^{-2})}f - f\rangle| \\ &\leq |A_{f,g}(x_0,\omega)| + \|g\|_2 \|L_{\Phi(x^2x_0^{-2})}f - f\|_2. \end{aligned}$$

There exists a neighbourhood $V(x_0) \subseteq G$ of x_0 such that

$$\|L_{\Phi(x^2x_0^{-2})}f - f\|_2 < \frac{\epsilon}{2\|g\|_2} \quad \text{for all } x \in V(x_0).$$

Since K is compact, it follows that we can choose finitely many elements $x_1, \dots, x_N \in G$, $N \in \mathbb{N}$, such that the sets $V(x_i)$, $i = 1, \dots, N$, cover K. Then we define $\Gamma \subset \widehat{G}$ by

$$\Gamma = \bigcup_{i=1}^{N} \Gamma(x_i).$$

Clearly, Γ is compact.

It remains to show that Γ satisfies the property mentioned above. Let $x \in K$. There exists $i_0 \in \{1, \dots, N\}$ such that $x \in V(x_{i_0})$. Hence, for all $\omega \in \widehat{G}\backslash\Gamma(x_{i_0})$,

$$|A_{f,g}(x,\omega)| = |A_{f,g}(x_{i_0},\omega)| + \|g\|_2 \|L_{\Phi(x^2x_{i_0}^{-2})}f - f\|_2 < \epsilon.$$

In particular, this is true for all $\omega \in \widehat{G}\backslash\Gamma$. Therefore, we obtain

$$|A_{f,g}(x,\omega)| < \epsilon \quad \text{for all } (x,\omega) \in (G \times \widehat{G})\backslash(K \times \Gamma).$$

Thus (i) holds.

The proof of (ii) being equivalent to (iii) follows the same main steps as the proof of (i) $\Leftrightarrow$ (iii) but we often have to use Lemma 2.1 (ii) in addition. □

3.3. **Square-integrability.** In this subsection we will establish necessary and sufficient conditions for a LCA group G to force any ambiguity function or Wigner distribution to be square-integrable. This problem can be simplified in an easy way.

For this, let $f, g \in L^2(G)$ and define the function $h_{f,g} : G \times G \to \mathbb{C}$ by

$$h_{f,g}(x,t) = \overline{f(t)}g(t\,\Phi(x^2)). \tag{1}$$

By Hölder's inequality, $h_{f,g}(x,\cdot) \in L^1(G)$ for each $x \in G$. Hence the Fourier transform of the function $t \mapsto h_{f,g}(x,t)$ exists and we have

$$\omega(\Phi(x))(h_{f,g}(x,\cdot))^\frown(\overline{\omega}) = A_{f,g}(x,\omega) \quad \text{for all } (x,\omega) \in G \times \widehat{G}.$$

Now suppose that $h_{f,g} \in L^2(G \times G)$. Then the Fourier transform and the Plancherel transform of $t \mapsto h_{f,g}(x,t)$ coincide and we obtain

$$\|A_{f,g}\|_2^2 = \|(x,\omega) \mapsto (h_{f,g}(x,\cdot))^\frown(\omega)\|_2^2 = \|h_{f,g}\|_2^2. \tag{2}$$

Conversely, suppose that $A_{f,g}$ is square-integrable. Then, for almost all $x \in G$, we have $(h_{f,g}(x,\cdot))^\frown \in L^2(\widehat{G})$. In particular, this is the Plancherel transform. This also implies (2). So we have proven the following lemma.

Lemma 3.8. *Let G be a LCA group and let $f, g \in L^2(G)$. Then the following conditions are equivalent.*

(i) $A_{f,g} \in L^2(G \times \widehat{G})$.
(ii) $h_{f,g} \in L^2(G \times G)$.

The following theorem shows that we may restrict our attention to ambiguity functions, since all results concerning square-integrability obtained for the ambiguity function holds for the Wigner distribution in the same manner.

Theorem 3.9. *Let G be a LCA group and let $f, g \in L^2(G)$. Then*

$$\|A_{f,g}\|_2 = \|W_{f,g}\|_2.$$

In particular, $A_{f,g}$ is square-integrable if and only if $W_{f,g}$ is square-integrable.

Proof. Consider the function $\tilde{h}_{f,g} : G \times G \to \mathbb{C}$ defined by

$$\tilde{h}_{f,g}(x,t) = \overline{f(x\Phi(t^{-1}))}g(x\Phi(t)).$$

Using Lemma 2.1 (i) and the same arguments as in the proof of Lemma 3.8 we obtain $\|W_{f,g}\|_2 = \|\tilde{h}_{f,g}\|_2$. Let $h_{f,g}$ be defined as in (1). In the proof of Lemma 3.8 we showed that $\|A_{f,g}\|_2 = \|h_{f,g}\|_2$. The claim now follows from

$$\|h_{f,g}\|_2^2 = \int_G \int_G |f(t)|^2 |g(t\Phi(x^2))|^2 \, dt \, dx \overset{t \mapsto t\Phi(x^{-1})}{=} \|\tilde{h}_{f,g}\|_2^2.$$

□

Therefore from now on we focus on the ambiguity function. By Lemma 3.8, in order to prove the square-integrability of an ambiguity function $A_{f,g}$, it suffices to show that $h_{f,g} \in L^2(G \times G)$. Notice that, for $G = \mathbb{R}$, it is not very difficult to conclude from Lemma 3.8 that $A_{f,g}$ is square-integrable for any two functions $f, g \in L^2(\mathbb{R})$ ([2, Theorem 2.1]). The case of an arbitrary LCA group is not so easy to deal with, since then $A_{f,g}$ is not always square-integrable. Here we shall characterize those LCA groups G which satisfies $A_{f,g} \in L^2(G \times \widehat{G})$ for all $f, g \in L^2(G)$.

For the proofs of the following theorems, we need some preparation. Let G be a LCA group. For the remainder of this subsection, we shall always denote the map $x \mapsto x^2$, $G \to G$ by φ. Furthermore, we define $\psi : G/\ker\varphi \to G^{(2)}$ by

$$\psi([x]) = x^2.$$

The function ψ has the following property.

Lemma 3.10. *Let G be a LCA group such that $G^{(2)}$ is closed. Then ψ is a topological isomorphism.*

Proof. Obviously, the mapping $x \mapsto x^2$, $G \to G^{(2)}$ is a surjective and continuous homomorphism. Hence, by [10, Theorem 5.29], it is also open, since $G^{(2)}$ is locally compact and G is σ-compact. Using [10, Theorem 5.27], the claim follows. □

Remark 3.11. There exist LCA groups such that ψ is not a topological isomorphism. For example, consider the group $H := \prod_{i=1}^{\infty} \mathbb{Z}_2$ endowed with the product topology. Then define G by $G = \prod_{i=1}^{\infty} \mathbb{Z}_4$ and regard H as a subgroup of G, where we identify the elements of $\mathbb{Z}_2$ with the elements of $\mathbb{Z}_4$ of order ≤ 2. We endow G with the topology such that H is an open and compact subgroup of G. Obviously, $\varphi(H) = \{e\}$ and $G^{(2)} = H$. Hence the map $x \mapsto x^2$, $G \to G^{(2)}$ is not open. Note that this map is open if and only if ψ is open. Thus here ψ is not a topological isomorphism.

Suppose that $G^{(2)}$ is closed. In this case we give a condition equivalent to the square-integrability of the ambiguity function. This condition is much easier to check.

Theorem 3.12. *Let G be a LCA group. Suppose that $G^{(2)}$ is closed. Then the following conditions are equivalent.*

(i) $A_{f,g} \in L^2(G \times \widehat{G})$ *for all* $f, g \in L^2(G)$.
(ii) $\ker \varphi$ *is compact and* $G^{(2)}$ *is open.*

Proof. In the following we will denote $\ker \varphi$ by K. By Lemma 3.10, ψ is a topological isomorphism. So, in particular, there exists a positive constant μ such that the Haar measure on G/K is equal to μ times the Haar measure on $G^{(2)}$. Without loss of generality, for the remainder of the proof, we can assume that the Haar measures on G, K, G/K, $G^{(2)}$ and $G/G^{(2)}$ are normalized so that for K and $G^{(2)}$ Weil's formula holds, respectively.

First, suppose that (i) holds. We claim that K is compact. Let $f, g \in L^2(G)$. By Weil's formula, we obtain

$$\infty > \int_{\widehat{G}} \int_G |A_{f,g}(x,\omega)|^2 \, dx \, d\omega = \int_{\widehat{G}} \int_{G/K} \left(\int_K |A_{f,g}(xk,\omega)|^2 \, dk \right) d(xK) \, d\omega.$$

This implies

$$\int_K |A_{f,g}(xk,\omega)|^2 \, dk < \infty \quad \text{for almost all } xK \in G/K, \omega \in \widehat{G}.$$

But on the other hand we have $|A_{f,g}(xk,\omega)| = |A_{f,g}(x,\omega)|$ for all $k \in K$. If K is non-compact, this implies $A_{f,g} = 0$ for all $f, g \in L^2(G)$, a contradiction.

It remains to prove that $G^{(2)}$ is open. Assume, towards a contradiction, that $G^{(2)}$ is not open. Using Lemma 2.1 (i), for $f, g \in L^2(G)$, we obtain

$$\|h_{f,g}\|_2^2 = \int_G |f(t)|^2 \int_G |g(t\Phi(x^2))|^2 dx dt \geq \frac{1}{c} \int_G |f(\Phi(t))|^2 \int_G |g(\Phi(tx^2))|^2 dx dt.$$

Observe that, if the function $t \mapsto |f(t)|^2 \int_G |g(t\Phi(x^2))|^2 \, dx$ is not integrable, we obtain a contradiction at once by Lemma 3.8. In the other case the requirements of Lemma 2.1 (i) are fulfilled. Now it suffices to prove the existence of functions $\tilde{f}, \tilde{g} \in L^2(G)$ such that

$$\int_G |\tilde{f}(t)|^2 \int_G |\tilde{g}(tx^2)|^2 \, dx \, dt = \infty.$$

Then we may define a function $f : G \to \mathbb{C}$ by

$$f(t) = \begin{cases} 0 & : \quad t \in G \backslash H, \\ \tilde{f}(\Phi^{-1}(t)) & : \quad t \in H \end{cases}$$

and a function $g : G \to \mathbb{C}$ in an analogous way. By Lemma 2.1 (ii), we obtain

$$\int_G |f(t)|^2 \, dt = \int_G |\chi_H(t) \tilde{f}(\Phi^{-1}(t))|^2 \, dt = \int_H |\tilde{f}(\Phi^{-1}(t))|^2 \, dt = \frac{1}{d} \int_G |\tilde{f}(t)|^2 \, dt,$$

where the Haar measure on H is induced by the Haar measure on G. Hence $f, g \in L^2(G)$. Thus we would obtain $\|h_{f,g}\|_2^2 = \infty$. By Lemma 3.8, this is a contradiction.

By assumption, $G^{(2)}$ is not open in G. Hence $G/G^{(2)}$ is not discrete. We can construct a function $h \in L^1(G/G^{(2)}) \backslash L^2(G/G^{(2)})$. Then, by [11, Theorem 28.54 (iii)], there exists a function $\tilde{h} \in L^1(G)$ such that

$$h(tG^{(2)}) = \int_{G^{(2)}} \tilde{h}(tx) \, dx.$$

Define $\tilde{f} : G \to \mathbb{R}$ by

$$\tilde{f}(y) = \sqrt{|\tilde{h}(y)|}.$$

Further, let $\tilde{g} := \tilde{f}$. Then, by using Weil's formula,

$$\begin{aligned} &\int_G |\tilde{f}(t)|^2 \int_G |\tilde{g}(tx^2)|^2 \, dx \, dt \\ &= \int_G |\tilde{f}(t)|^2 \int_{G/K} \int_K |\tilde{f}(t(yh)^2)|^2 \, dh \, d(yK) \, dt \\ &= |K| \int_G |\tilde{f}(t)|^2 \int_{G/K} |\tilde{f}(ty^2)|^2 \, d(yK) \, dt \\ &= \mu |K| \int_G |\tilde{f}(t)|^2 \int_{G^{(2)}} |\tilde{f}(ty)|^2 \, dy \, dt. \end{aligned}$$

If the map $t \mapsto |\tilde{f}(t)|^2 \int_{G^{(2)}} |\tilde{f}(ty)|^2 \, dy$ does not belong to $L^1(G)$, we are done. Otherwise, using Weil's formula again, we obtain

$$\begin{aligned}
&\int_G |\tilde{f}(t)|^2 \int_G |\tilde{g}(tx^2)|^2\,dx\,dt \\
&= \mu|K| \int_{G/G^{(2)}} \int_{G^{(2)}} |\tilde{f}(th)|^2 \int_{G^{(2)}} |\tilde{f}(thy)|^2\,dy\,dh\,d(tG^{(2)}) \\
&= \mu|K| \int_{G/G^{(2)}} \left[\int_{G^{(2)}} |\tilde{f}(th)|^2\,dh\right]^2 d(tG^{(2)}) \\
&= \mu|K| \int_{G/G^{(2)}} \left[\int_{G^{(2)}} |\tilde{h}(th)|\,dh\right]^2 d(tG^{(2)}) \\
&\geq \mu|K| \int_{G/G^{(2)}} |h(tG^{(2)})|^2\,d(tG^{(2)}) \\
&= \infty.
\end{aligned}$$

As mentioned above, this is a contradiction. Thus we proved (i) $\Rightarrow$ (ii).

Now suppose that (ii) holds. Let $f, g \in L^2(G)$. Using Weil's formula, we obtain

$$\begin{aligned}
\|h_{f,g}\|_2^2 &= \int_G |f(t)|^2 \int_G |g(t\,\Phi(x^2))|^2\,dx\,dt \\
&= \int_G |f(t)|^2 \int_{G/K} \int_K |g(t\,\Phi((yh)^2))|^2\,dh\,d(yK)\,dt \\
&= |K| \int_G |f(t)|^2 \int_{G/K} |g(t\,\Phi(y^2))|^2\,d(yK)\,dt.
\end{aligned}$$

Since $G^{(2)}$ is open, there exists a positive constant ν such that the Haar measure on $G^{(2)}$ is ν times the measure on $G^{(2)}$ induced by the Haar measure on G. Thus, also using Lemma 2.1 (i), we obtain

$$\begin{aligned}
\|h_{f,g}\|_2^2 &= \mu|K| \int_G |f(t)|^2 \int_{G^{(2)}} |g(t\,\Phi(x))|^2\,dx\,dt \\
&= \mu\nu|K| \int_G |f(t)|^2 \int_G \chi_{G^{(2)}}(x)|g(t\,\Phi(x))|^2\,dx\,dt \\
&\leq \mu\nu|K| \int_G |f(t)|^2 \int_G |g(t\,\Phi(x))|^2\,dx\,dt \\
&\leq \mu\nu c|K| \int_G |f(t)|^2 \int_G |g(tx)|^2\,dx\,dt \\
&= \mu\nu c|K|\|f\|_2^2\|g\|_2^2.
\end{aligned}$$

Since, by hypothesis, K is compact, we have shown $h_{f,g} \in L^2(G \times G)$. Now Lemma 3.8 yields (i). □

The condition that $G^{(2)}$ be closed is not very restrictive as is shown in the next proposition.

Proposition 3.13. *Let G be a 2-root compact LCA group. Then $G^{(2)}$ is closed.*

Proof. We argue by contradiction and assume that $G^{(2)}$ is not closed. This means that there exist $y \in G \backslash G^{(2)}$ and a net $(x_\iota)_{\iota \in I}$ in G such that $x_\iota^2 \to y$. Let K be a compact neighbourhood of y. Then $x_\iota^2 \in K$ for all $\iota \geq \iota_0$. Notice that the set $\{x_\iota^2 : \iota \geq \iota_0\} \cup \{y\}$ is compact. Since G is 2-root compact, $\varphi^{-1}(\{x_\iota^2 : \iota \geq \iota_0\} \cup \{y\})$ is also compact. Hence there exists a convergent subnet $(x_{\iota_\lambda})_{\lambda \in I}$ of $(x_\iota)_{\iota \in I}$. Let $x \in G$ be defined by $x_{\iota_\lambda} \to x$. By continuity of φ, we obtain $y = x^2 \in G^{(2)}$, a contradiction. □

The preceding proposition and [12, Example 3.1.3] show that Theorem 3.12 applies, for instance, to all compactly generated LCA groups.

Under slightly stronger conditions than that $G^{(2)}$ is closed, we can find necessary and sufficient conditions for the ambiguity functions to be square-integrable, which are easier to check. However, by Theorem 3.12, square-integrability of all ambiguity functions implies that $G^{(2)}$ is open. Thus the requirement that $G^{(2)}$ has to be open is not very restrictive.

Theorem 3.14. *Let G be a LCA group. Suppose that $G^{(2)}$ is open. Then the following conditions are equivalent.*

(i) $A_{f,g} \in L^2(G \times \widehat{G})$ *for all* $f, g \in L^2(G)$.
(ii) $\ker \varphi$ *is compact.*
(iii) G *is 2-root compact.*

Proof. The equivalence of the statements (i) and (ii) follows from Theorem 3.12. Suppose that (iii) holds. Then $\ker \varphi = \varphi^{-1}(\{e\})$ is compact. This implies (ii). To finish the proof we show that (ii) implies (iii). Suppose that $G^{(2)}$ is open and $\ker \varphi$ is compact. Since ψ is a topological isomorphism, $\psi^{-1}(K) \subseteq G/\ker \varphi$ is compact for all compact subsets $K \subseteq G^{(2)}$. Since $\ker \varphi$ is compact, (iii) follows immediately. □

The next corollary shows that a large class of LCA groups satisfy the hypothesis of Theorem 3.14. However, first we want to exhibit a property for a LCA group G which forces $G^{(2)}$ to be open.

Proposition 3.15. *Let G be a LCA group. Suppose that G_0 is open. Then $G^{(2)}$ is open. In particular, $G^{(2)}$ is open for all LCA Lie groups.*

Proof. Suppose that G_0 is open. By the structure theorem for LCA groups [10, Theorem 24.30], there exist a compact abelian group C and $p \geq 0$ such that $\mathbb{R}^p \times C$ is an open subgroup of G. We have $G_0 \subseteq \mathbb{R}^p \times C_0$. Since G_0 is open, also $\mathbb{R}^p \times C_0$ is open. Recall that compact, connected abelian groups are divisible ([10, Theorem 24.25]). Thus we have $C_0 \subseteq C^{(2)}$. This yields

$$\mathbb{R}^p \times C_0 \subseteq \mathbb{R}^p \times C^{(2)} \subseteq G^{(2)}.$$

Hence $G^{(2)}$ is open. □

Corollary 3.16. *Let G be a 2-root compact LCA group. Suppose that G_0 is open. Then, for all $f, g \in L^2(G)$,*

$$A_{f,g} \in L^2(G \times \widehat{G}).$$

Proof. By Proposition 3.15, $G^{(2)}$ is open. Then the claim follows from Theorem 3.14. □

In particular, the ambiguity function is always square-integrable for all elementary LCA groups.

The next proposition gives an estimate for the norm of the ambiguity function, which will be needed in Subsection 3.4.

Proposition 3.17. *Let G be a LCA group. Suppose that $G^{(2)}$ is closed and that $A_{f,g} \in L^2(G \times \widehat{G})$ for all $f, g \in L^2(G)$.*

(i) *There exists a positive constant C such that, for all $f, g \in L^2(G)$,*

$$\|A_{f,g}\|_2^2 \leq C\|f\|_2^2\|g\|_2^2.$$

(ii) *The mapping*

$$(f,g) \mapsto A_{f,g}, \quad L^2(G) \times L^2(G) \to L^2(G \times \widehat{G})$$

is continuous.

Proof. Let $f, g \in L^2(G)$. Using the proof of Lemma 3.8, we obtain $\|A_{f,g}\|_2^2 = \|h_{f,g}\|_2^2$. By Theorem 3.12 ((i) $\Rightarrow$ (ii)), $\ker\varphi$ is compact and $G^{(2)}$ is open. Hence there exist suitable positive constants μ, ν and c such that

$$\|h_{f,g}\|_2^2 \leq \mu\nu c|\ker\varphi|\|f\|_2^2\|g\|_2^2$$

as was shown in the proof of Theorem 3.12 ((ii) $\Rightarrow$ (i)) and we have $|\ker\varphi| < \infty$. This proves (i).

The claim in (ii) is a direct conclusion from (i). □

3.4. $\widehat{A_{f,g}} = W_{f,g}$**.** In this subsection we will show that the Plancherel transform of the ambiguity function coincides with the Wigner distribution for a large class of LCA groups.

We need some preparations before we give the theorem and its proof. The following theorem is [11, Theorem 31.13], where we chose $\Delta = G$.

Theorem 3.18. [11, Theorem 31.13] *Let G be a LCA group. Let $C_0^+(G)$ and $C_0^+(\widehat{G})$ denote the space of functions which have only positive values and which belong to $C_0(G)$ and $C_0(\widehat{G})$, respectively. Then there exist sequences $(k_n)_{n\in\mathbb{N}} \subseteq C_0^+(\widehat{G}) \cap L^1(\widehat{G})$ and $(\psi_n)_{n\in\mathbb{N}} \subseteq C_0^+(G) \cap L^1(G)$ such that for all $n \in \mathbb{N}$, $\omega \in \widehat{G}$, $x \in G$ and $f \in C_c(G)$ the following is true.*

(i) $k_n(\widehat{G}) \subseteq [0,1]$ *and* $k_n = k_n^*$.

(ii) $\lim_{n\to\infty} k_n(\omega) = 1$.

(iii) $\check{k}_n = \psi_n$, *where* ˇ *denotes the inverse Fourier transform.*

(iv) $\int_G \psi_n(x)\,dx = 1$ *and* $\psi_n = \psi_n^*$.

(v) $\lim_{n\to\infty} (\omega * \psi_n)(x) = \omega(x)$.

(vi) $\lim_{n\to\infty} (f * \psi_n)(x) = f(x)$.

The next two lemmas will be used in the proof of the theorem.

Lemma 3.19. *Let G be a 2-root compact LCA group. Let $f, g \in C_c(G)$ and define the function $\tilde{h}_{f,g} : G \times G \to \mathbb{C}$ (as in the proof of Theorem 3.9) by*

$$\tilde{h}_{f,g}(x,t) = \overline{f(x\Phi(t^{-1}))}g(x\Phi(t)).$$

Let $(\psi_n)_{n\in\mathbb{N}}$ be as in Theorem 3.18. Then

(i) *$\tilde{h}_{f,g} \in C_c(G \times G)$.*

(ii) *For each $x \in G$, $(\tilde{h}_{f,g}(\cdot,t) * \overline{\psi}_n)(x)$ converges uniformly in $t \in G$ to $\tilde{h}_{f,g}(x,t)$.*

Proof. Clearly, $\tilde{h}_{f,g}$ is continuous. Define $T \subseteq G$ by $T = \text{supp} f \cup \text{supp} g$. Note that T is compact. Furthermore, define $C_1, C_2 \subseteq G$ by

$$C_1 = \{t \in G : t^2 \in \Phi^{-1}(T^{-1}T)\} \quad \text{and} \quad C_2 = \Phi(C_1)T \cap \Phi(C_1)^{-1}T.$$

Obviously, C_1 and C_2 are also compact, since G is supposed to be 2-root compact. It is straightforward to check that supp $\tilde{h}_{f,g} \subseteq C_2 \times C_1$. Thus (i) holds.

Using Theorem 3.18 (vi), this implies immediately that the convergence is uniform. □

Lemma 3.20. *Let G be a 2-root compact LCA group. Further, let $f, g \in C_c(G)$ and let $(k_n)_{n\in\mathbb{N}}$ be as in Theorem 3.18. Then, for all $n \in \mathbb{N}$,*

$$A_{f,g} \cdot k_n \in L^1(G \times \widehat{G}).$$

Proof. Let $n \in \mathbb{N}$. By Theorem 3.18 (iii), Lemma 3.19 (i) and using the fact that $(k_n)_{n\in\mathbb{N}} \subseteq C_0^+(\widehat{G}) \cap L^1(\widehat{G})$, we obtain

$$\begin{aligned}
&\int_{G\times\widehat{G}} |A_{f,g}(x,\omega)k_n(\omega)|\, d(x,\omega) \\
&\leq \int_{\widehat{G}}\int_G\int_G |f(t\Phi(x^{-1}))||g(t\Phi(x))|\, dt\, k_n(\omega)\, dx\, d\omega \\
&= \int_G\int_G\int_{\widehat{G}} k_n(\omega)\, d\omega |f(t\Phi(x^{-1}))||g(t\Phi(x))|\, dx\, dt \\
&= \psi_n(e)\int_G\int_G |\tilde{h}_{f,g}(t,x)|\, dx\, dt \\
&< \infty.
\end{aligned}$$

This proves the lemma. □

For $G = \mathbb{R}$, the following result is contained in [5, Section 2] but without a detailed proof.

Theorem 3.21. *Let G be a 2-root compact LCA group. Suppose that $G^{(2)}$ is open. Then, for all $f, g \in L^2(G)$,*

$$\widehat{A_{f,g}} = W_{f,g} \quad \text{in } L^2(\widehat{G} \times G).$$

Proof. Theorem 3.14 implies that $A_{f,g} \in L^2(G \times \widehat{G})$ for all $f, g \in L^2(G)$.

By Proposition 3.17 (i), by the corresponding result for the Wigner distribution and by the Plancherel theorem, it suffices to prove that $\widehat{A_{f,g}} = W_{f,g}$ for $f, g \in C_c(G)$.

For this, let $(k_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ be as in Theorem 3.18. Since $k_n \in C_0(\widehat{G})$, the function

$$(x,\omega) \mapsto A_{f,g}(x,\omega)k_n(\omega), \quad G \times \widehat{G} \to \mathbb{C}$$

is square-integrable. Hence the Plancherel transforms of both $A_{f,g}$ and $A_{f,g}\cdot k_n$ exist. By Theorem 3.18 (ii), for all $(x,\omega) \in G \times \widehat{G}$,

$$\lim_{n\to\infty} A_{f,g}(x,\omega)k_n(\omega) = A_{f,g}(x,\omega).$$

In addition, by Theorem 3.18 (i), we obtain

$$|A_{f,g}(x,\omega)k_n(\omega) - A_{f,g}(x,\omega)|^2 = |k_n(\omega)-1|^2|A_{f,g}(x,\omega)|^2 \leq |A_{f,g}(x,\omega)|^2$$

for all $(x,\omega) \in G \times \widehat{G}$. Then, by the theorem of dominated convergence,

$$\lim_{n\to\infty} \|A_{f,g}\cdot k_n - A_{f,g}\|_2 = 0,$$

and hence, by the Plancherel theorem,

$$\lim_{n\to\infty} \|\widehat{A_{f,g}\cdot k_n} - \widehat{A_{f,g}}\|_2 = 0.$$

This implies that it suffices to prove

$$\lim_{n\to\infty} \widehat{A_{f,g}\cdot k_n}(\omega,x) = W_{f,g}(\omega,x)$$

for almost all $(\omega,x) \in \widehat{G} \times G$.

For this, notice that the Plancherel transform of $A_{f,g}\cdot k_n$ coincides with the Fourier transform by Lemma 3.20. Then, for almost all $(\omega,x) \in \widehat{G} \times G$,

$$\begin{aligned}
\widehat{A_{f,g}\cdot k_n}(\omega,x) &= \int_G \int_{\widehat{G}} A_{f,g}(t,\chi)k_n(\chi)\overline{\omega(t)}\,\overline{\chi(x)}\,d\chi\,dt \\
&= \int_G \int_{\widehat{G}} \int_G \overline{f(y\Phi(t^{-1}))}g(y\Phi(t))\chi(y)\,dy k_n(\chi)\overline{\omega(t)}\,\overline{\chi(x)}\,d\chi\,dt \\
&= \int_G \int_G \int_{\widehat{G}} k_n(\chi)\chi(yx^{-1})\,d\chi \overline{f(y\Phi(t^{-1}))}g(y\Phi(t))\overline{\omega(t)}\,dy\,dt.
\end{aligned}$$

Since $k_n \in L^1(\widehat{G})$ as well as $fg \in L^1(G)$, we are allowed to use Fubini's theorem in the last step. Then, by Theorem 3.18 (iii), (iv),

$$\begin{aligned}
\widehat{A_{f,g}\cdot k_n}(\omega,x) &= \int_G \int_G \psi_n(yx^{-1})\overline{f(y\Phi(t^{-1}))}g(y\Phi(t))\overline{\omega(t)}\,dy\,dt \\
&= \int_G ((\overline{f(\cdot\Phi(t^{-1}))}g(\cdot\Phi(t))) * \overline{\psi}_n)(x)\overline{\omega(t)}\,dt.
\end{aligned}$$

Hence, using Theorem 3.18 (vi) and Lemma 3.19 (ii), we obtain, for all $(\omega, x) \in \widehat{G} \times G$,

$$\begin{aligned} W_{f,g}(\omega, x) &= \int_G \overline{f(x\Phi(t^{-1}))}g(x\Phi(t))\overline{\omega(t)}\,dt \\ &= \int_G \lim_{n\to\infty}((\overline{f(\cdot\Phi(t^{-1}))}g(\cdot\Phi(t))) * \overline{\psi}_n)(x)\overline{\omega(t)}\,dt \\ &= \lim_{n\to\infty}\int_G ((\overline{f(\cdot\Phi(t^{-1}))}g(\cdot\Phi(t))) * \overline{\psi}_n)(x)\overline{\omega(t)}\,dt \\ &= \lim_{n\to\infty} \widehat{A_{f,g}\cdot k_n}(\omega, x). \end{aligned}$$

This shows

$$\widehat{A_{f,g}} = W_{f,g} \quad \text{for all } f, g \in C_c(G). \qquad \square$$

By Theorem 3.12, the condition that $G^{(2)}$ is supposed to be open is necessary for $A_{f,g}$ to be square-integrable.

4. Generalization of Cohen's class

In analogy with Cohen [6] (compare also [9, Subsection 4.5]), we may define a general class of time-frequency representations for LCA groups. For this, let G be a LCA group and let $\phi : G \times \widehat{G} \to \mathbb{C}$ be a function, which we will call *kernel*. Then depending on a signal $f \in L^2(G)$ we may consider the function $C_\phi f : \widehat{G} \times G \to \mathbb{C}$ defined by

$$C_\phi f = \widehat{A_f \cdot \phi},$$

whenever $A_f \cdot \phi \in L^2(G \times \widehat{G})$. As for $G = \mathbb{R}$, an abundance of time-frequency representations of G can be constructed this way. Note that, if G is 2-root compact and $G^{(2)}$ is open, we have $C_\phi f = W_f$ for $\phi \equiv 1$ by Theorem 3.21.

This method has a great advantage, since we are able to check in advance important properties of the time-frequency representation constructed in this way by only knowing the kernel. In the following we state how the main properties of $C_\phi f$ depend on the kernel.

Theorem 4.1. *Let G be a 2-root compact LCA group such that $G^{(2)}$ is open, ϕ a kernel such that $A_g \cdot \phi \in L^2(G \times \widehat{G})$ for all $g \in L^2(G)$ and let $f \in L^2(G)$. Then the following hold.*

(i) *Marginals:*

$$\int_{\widehat{G}} C_\phi f(\omega, x)\,d\omega = |f(x)|^2 \text{ for all } x \in G \Leftrightarrow \phi(e, \cdot) \equiv 1.$$

$$\int_G C_\phi f(\omega, x)\,dx = ((f^* * f) \circ \Phi^2)^\wedge(\omega) \text{ for all } \omega \in \widehat{G} \Leftrightarrow \phi(\cdot, 1) \equiv 1.$$

(ii) *Total energy:*

$$\int_G \int_{\widehat{G}} C_\phi f(\omega, x)\, d\omega dx = \|f\|_2^2 \Leftrightarrow \phi(e,1) = 1.$$

(iii) *Reality:*

$$C_\phi f \text{ is real } \Leftrightarrow \phi = \phi^*.$$

(iv) *Shift invariance:*

$$C_\phi f(\omega, x) \text{ is time-shift invariant } \Leftrightarrow \phi \text{ is independent of } x.$$

$$C_\phi f(\omega, x) \text{ is frequency-shift invariant } \Leftrightarrow \phi \text{ is independent of } \omega.$$

In our general case shift invariance means $C_\phi(\omega_0 \cdot L_{x_0^{-1}} f)(\omega, x) = C_\phi f(\omega(\overline{\omega_0}^2 \circ \Phi), xx_0)$.

Proof. The main steps of the proof parallel those for $G = \mathbb{R}$ ([7, Subsection 9.4] and [9, Subsection 4.5]). Here, in addition, we have to use arguments similar to those in the proof of Theorem 3.21. □

Example 4.2. We consider the case $G = \mathbb{R}$. Note that all relations reduce to the well-known relations when $H = \mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$ is defined by $\Phi(x) = \frac{x}{2}$. Now let $H = \mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$ be defined by $\Phi(x) = \alpha x$, $\alpha \in \mathbb{R}\backslash\{0\}$. Then the frequency-marginal takes the form

$$\int_{\mathbb{R}} C_\phi f(y, x)\, dx = \frac{1}{2\alpha} \left| \hat{f}\left(\frac{y}{2\alpha}\right) \right|^2 \text{ for all } y \in \mathbb{R} \Leftrightarrow \phi(\cdot, 1) \equiv 1,$$

hence yields a dilation of $\hat{f}$ on the right-hand side.

5. Applications to the group of p-adic numbers

Let p be a prime and let Ω_p denote the group of p-adic numbers. If we endow Ω_p with the usual addition [10, Definition 10.2] and topology [10, Theorem 10.5], it becomes a LCA group.

A Haar measure on Ω_p can be constructed in the following way. Let ν be the normalized counting measure on $\{0, 1, \dots, p-1\}$. For each $n \in \mathbb{Z}$, let μ_n denote the corresponding product measure on the space $\prod_{k=n}^{\infty}\{0, 1, \dots, p-1\}$. Let $\Lambda_n := \{y \in \Omega_p : y_k = 0 \text{ for } k < n\}$, $n \in \mathbb{Z}$. For a subset A of Ω_p, we define

$$\lambda(A) = \lim_{n\to\infty} p^n \mu_n(A \cap \Lambda_n).$$

We normalize λ by requiring that $\lambda(\Lambda_0) = 1$.

In addition, the group of p-adic numbers is selfdual, that means $\widehat{\Omega_p} = \Omega_p$. A topological isomorphism $y \mapsto \chi_y$, $\Omega_p \to \widehat{\Omega_p}$ can be defined as follows. If $y = 0$, then $\chi_y = 1$. Now assume that $y \neq 0$. Suppose that $y_n = 0$ for $n \leq k$ and $y_{k+1} \neq 0$. Then define

$$\lambda_n := \sum_{j=k+1}^{n} \frac{y_j}{p^{n-j+1}} \text{ for } n > k.$$

Let $x \in \Omega_p$. If $x \in \Lambda_{-k}$, then $\chi_y(x) = 1$. Otherwise, there exists an integer $m < -k$ such that $x_n = 0$ for $n < m$ and $x_m \neq 0$. Then

$$\chi_y(x) = e^{2\pi i(x_m\lambda_{-m}+x_{m+1}\lambda_{-m-1}+\ldots+x_{-k-1}\lambda_{k+1})}.$$

First, we examine Ω_p with respect to possible choices for H and Φ. By [10, 10.16 (a)], the only proper closed subgroups of Ω_p are the subgroups Λ_k. All subgroups Λ_k are even open by the definition of the topology of Ω_p. Hence the only possible choice for H is $H = \Omega_p$ or $H = \Lambda_k$, $k \in \mathbb{Z}$. But since the subgroups Λ_k are compact and Ω_p is non-compact, Λ_k is not topologically isomorphic to Ω_p for each $k \in \mathbb{Z}$. Thus H has to be chosen to equal Ω_p. Then all possible automorphisms Φ are of the form (compare [10, Example 26.18 (d)])

$$\Phi : \Omega_p \to \Omega_p,\ \Phi(x) = ax \text{ for some } a \in {\Omega_p}^*.$$

The definition of the multiplication can be found in [10, Definition 10.9].

Thus the ambiguity function and Wigner distribution of some functions $f, g \in L^2(\Omega_p)$ are of the form

$$A_{f,g}(x,y) = \int_{\Omega_p} \overline{f(t-ax)}g(t+ax)\chi_y(t)\,dt$$

and

$$W_{f,g}(y,x) = \int_{\Omega_p} \overline{f(x-at)}g(x+at)\overline{\chi_y(t)}\,dt.$$

Next we are going to prove that ${\Omega_p}^{(2)}$ is open. For this, let the map $x \mapsto x^2$, $\Omega_p \to \Omega_p$ be denoted by φ and let $k \in \mathbb{Z}$. Since $\varphi(\Lambda_k)$ is a compact, hence closed subgroup of Ω_p, [10, 10.16 (a)] implies that $\varphi(\Lambda_k) = \Lambda_l$ for some $l \in \mathbb{Z}$. It is clear that, for each $m \in \mathbb{Z}$, we construct $y \in \Lambda_m$ and $x \in \Omega_p$ such that $\varphi(x) = y$. Thus we obtain ${\Omega_p}^{(2)} = \Omega_p$. Hence the claim is proven.

Ω_p is also 2-root compact. We prove this by showing that in this case φ is a topological isomorphism. It is easy to check that φ is bijective. It remains to prove that φ is open. The proof of ${\Omega_p}^{(2)} = \Omega_p$ implies that, for each $k \in \mathbb{Z}$, there exists some $l \in \mathbb{Z}$ such that $\varphi(\Lambda_k) = \Lambda_l$. Hence, by the definition of the topology of Ω_p, φ is open.

Thus the hypotheses of Theorem 3.7, Theorem 3.14 and the corresponding result for the Wigner distribution and Theorem 3.21 are fulfilled. This implies the following theorem.

Theorem 5.1. *Let $f, g \in L^2(\Omega_p)$. Then the following hold.*

(i) $A_{f,g} \in C_0(\Omega_p \times \widehat{\Omega_p})$ *and* $W_{f,g} \in C_0(\widehat{\Omega_p} \times \Omega_p)$.
(ii) $A_{f,g} \in L^2(\Omega_p \times \widehat{\Omega_p})$ *and* $W_{f,g} \in L^2(\widehat{\Omega_p} \times \Omega_p)$.
(iii) $\widehat{A_{f,g}} = W_{f,g}$.

Let $f \in L^2(\Omega_p)$ and let $\phi : \Omega_p \times \widehat{\Omega_p} \to \mathbb{C}$ be some function such that $A_f \cdot \phi \in L^2(\Omega_p \times \widehat{\Omega_p})$. Then the time-frequency representations for the group of p-adic numbers which belong to Cohen's class can be obtained from

$$C_\phi f(y,x) = \int_{\Omega_p}\int_{\widehat{\Omega_p}}\int_{\Omega_p} \overline{f(s-at)} f(s+at)\chi_r(s)\, ds\, \phi(t,r)\overline{\chi_y(t)}\,\overline{\chi_r(x)}\, drdt.$$

Acknowledgment

The author is indebted to the referee for some useful suggestions.

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