BEURLING DENSITY AND SHIFT-INVARIANT WEIGHTED IRREGULAR GABOR SYSTEMS

GITTA KUTYNIOK

Abstract. In this paper we introduce and study a concept to assign a shift-invariant weighted Gabor system to an irregular Gabor system while preserving special properties such as being a frame. First we extend the notion of Beurling density to weighted subsets of $\mathbb{R}^d$. We then derive a useful reinterpretation of this definition by using arbitrary piecewise continuous, positive functions in the amalgam space $W(L^\infty, L^1)$ to measure weighted Beurling density, thereby generalizing a result by Landau in the non-weighted situation. Using, in addition, decay properties of the short-time Fourier transform of functions contained in the modulation space $M^1(\mathbb{R}^d)$, we establish a fundamental relationship between the weighted Beurling density of the set of indices, the frame bounds, and the norm of the generator for weighted Gabor frames. Finally, we prove that the relation among an irregular Gabor system and its shift-invariant counterpart imposes special conditions on the weighted Beurling densities of their sets of indices.

1. Introduction

Shift-invariance, i.e., invariance under integer translations, is a desirable feature for many applications. Moreover, shift-invariance has been extensively employed as a theoretical tool to study regular Gabor systems $\{e^{2\pi i\langle x,\beta n\rangle}g(x-\alpha k) : k,n \in \mathbb{Z}^d\}$, $g \in L^2(\mathbb{R}^d)$, $\alpha, \beta > 0$ (see, for example, [10, 14]). In this paper we will study this concept in a much more general setting motivated by the following observation. Provided that $\{e^{2\pi i\langle x,\beta n\rangle}g(x-\alpha k) : k,n \in \mathbb{Z}^d\}$ forms a frame for $L^2(\mathbb{R}^d)$, then each $f \in L^2(\mathbb{R}^d)$ can be reconstructed in a numerically stable way from the sampling points $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ of the short-time Fourier transform $V_g f : \mathbb{R}^{2d} \to \mathbb{C}$, $V_g f(a,b) = \left\langle f, e^{2\pi i\langle \cdot,b\rangle}g(\cdot - a)\right\rangle$. However, sampling points may vary in practice, and the question arises whether and how substituting the subset $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ by an arbitrary subset Λ of $\mathbb{R}^{2d}$ affects results concerning shift-invariance. Therefore, our driving motivation is to introduce a concept which assigns a shift-invariant Gabor system to an arbitrary irregular Gabor system while preserving special properties; in this sense capturing the spirit of the relation between affine and quasi-affine systems (see, for instance, [4]). By considering the similar situation in wavelet analysis it becomes evident that the shift-invariant counterpart necessarily has to be equipped with weights. Therefore the construction of such associated shift-invariant Gabor systems leads to the introduction of more general weighted irregular Gabor systems

$$\mathcal{G}(g,\Lambda,w) := \{w(a,b)^{\frac{1}{2}}e^{2\pi i\langle x,b\rangle}g(x-a) : (a,b) \in \Lambda\},$$

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where $g \in L^2(\mathbb{R}^d)$, $\Lambda \subset \mathbb{R}^{2d}$, and $w : \Lambda \to \mathbb{R}^+$.

Density conditions have turned out to be an especially useful and elegant tool for the study of irregular Gabor systems [2, 3, 13]. The densities employed for Gabor systems are the lower and upper Beurling densities, which are defined for an arbitrary countable subset of $\mathbb{R}^d$. Therefore, in order to study the relationship between an irregular Gabor system $\mathcal{G}(g, \Lambda)$ and its associated shift-invariant counterpart $\mathcal{G}(g, \Lambda_{SI}, w)$, where Λ_{SI} is in some sense the "smallest" shift-invariant set containing Λ, we will focus on the conditions which are imposed on the densities of both systems.

We first show that Beurling density can not only be measured by computing the average over the number of points of a set contained in cubes $x + h[0, 1)^d$, $x \in \mathbb{R}^d$, $h > 0$, but also by computing the weighted sum over those points with respect to an arbitrary piecewise continuous, positive function in the amalgam space $W(L^\infty, L^1)$. In fact, we show that for each piecewise continuous $f \in W(L^\infty, L^1)$ with $f \geq 0$, $f \neq 0$, the upper Beurling density $\mathcal{D}^+(\Lambda)$ of a set $\Lambda \subset \mathbb{R}^d$ satisfies

$$\mathcal{D}^+(\Lambda) = \|f\|_1^{-1} \limsup_{h\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\sum_{\lambda\in\Lambda} f(\frac{\lambda+x}{h})}{h^d}$$

and the lower Beurling density satisfies a similar formula, thereby generalizing a result by Landau [12], which established this property for the functions $f = \chi_B$, where $B \subset \mathbb{R}^d$ is a compact set of measure 1 whose boundary has measure zero. Our result even holds for a generalization of Beurling density to weighted subsets. In particular, it implies that for functions contained in the modulation space $M^1(\mathbb{R}^d)$ the associated short-time Fourier transform can serve as a measuring function for the (weighted) Beurling densities.

By employing this new interpretation of (weighted) Beurling density and the decay properties of the short-time Fourier transform of M^1-functions, we will prove that the fundamental relationship

$$A \leq \mathcal{D}^-(\Lambda, w)\, \|g\|_2^2 \leq \mathcal{D}^+(\Lambda, w)\, \|g\|_2^2 \leq B \tag{1}$$

between the weighted Beurling densities $\mathcal{D}^-(\Lambda, w), \mathcal{D}^+(\Lambda, w)$ of the set of indices Λ with weight function w, the frame bounds A, B, and the norm of the generator g holds for weighted Gabor frames $\mathcal{G}(g, \Lambda, w)$. This result can be shown to contain [2, Thm. 4.2] and [5, Subsec. 3.4.1] as special cases. As a corollary we obtain that the weighted Beurling density of a tight weighted Gabor frame necessarily has to be uniform, i.e., $\mathcal{D}^-(\Lambda, w) = \mathcal{D}^+(\Lambda, w)$. Recently, some results in the same spirit have been derived for a class of wavelet frames [11] and for frames of windowed exponentials [9].

The relationship (1) then leads to constraints imposed on the weighted densities of a Gabor system and its associated shift-invariant weighted Gabor system. In particular, we consider sets $\Lambda \subset \mathbb{R}^{2d}$ and weight functions $w : \Lambda_{SI} \to \mathbb{R}^+$, which satisfy that there exists a function $g \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(g, \Lambda)$ and $\mathcal{G}(g, \Lambda_{SI}, w)$ are tight frames with the same frame bound. We prove that this hypothesis already implies that both Λ and (Λ_{SI}, w) have uniform density and

$$\mathcal{D}(\Lambda) = A\, \|g\|_2^{-2} = \mathcal{D}(\Lambda_{SI}, w). \tag{2}$$

Finally, we show that for a given $\Lambda \subset \mathbb{R}^{2d}$ with positive lower and finite upper density there always exists a weight function $w : \Lambda_{SI} \to \mathbb{R}^+$ which satisfies (2).

This paper is organized as follows. In Section 2 we introduce some notion and state some preliminary results. In Section 3 we first study relations between certain Littlewood–Paley type inequalities and density conditions. These are employed in Subsection 3.2 to derive a useful reinterpretation of weighted Beurling density (Theorem 3.5). Finally, in Section 4 we study shift-invariant weighted Gabor systems. The fundamental relationship for weighted Gabor frames between the weighted Beurling density, the frame bounds, and the norm of the generator (Theorem 4.2) is established in Subsection 4.1. We then give a precise definition of an associated shift-invariant weighted Gabor system (Subsection 4.2). Finally, in Subsection 4.3 we prove necessary density conditions for the sets of indices of a Gabor system and its shift-invariant counterpart (Theorem 4.11).

2. Notation and preliminary results

2.1. **Weighted Beurling density.** Beurling density is a measure of the "average" number of points of a set that lie inside a unit cube. Here we extend this notion to the situation of weighted subsets of $\mathbb{R}^d$.

For $h > 0$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we let $Q_h(x)$ denote the cube centered at x with side length h, i.e., $Q_h(x) = \prod_{j=1}^d \left[x_j - \frac{h}{2}, x_j + \frac{h}{2}\right)$. If $x = 0$, we often just write Q_h instead of $Q_h(0)$. Given a set Λ of points in $\mathbb{R}^d$ and a weight function $w : \Lambda \to \mathbb{R}^+$, we define the weighted number of elements of Λ lying in a subset K of Λ to be $\#_w(K) = \sum_{x \in K} w(x)$. Then the *upper weighted (Beurling) density* of the pair (Λ, w) is defined by

$$\mathcal{D}^+(\Lambda, w) = \limsup_{h \to \infty} \frac{\sup_{x \in \mathbb{R}^d} \#_w(\Lambda \cap Q_h(x))}{h^d},$$

and the *lower weighted (Beurling) density* of (Λ, w) is

$$\mathcal{D}^-(\Lambda, w) = \liminf_{h \to \infty} \frac{\inf_{x \in \mathbb{R}^d} \#_w(\Lambda \cap Q_h(x))}{h^d}.$$

If $\mathcal{D}^-(\Lambda, w) = \mathcal{D}^+(\Lambda, w)$, then we say that (Λ, w) has *uniform weighted (Beurling) density* and denote this density by $\mathcal{D}(\Lambda, w)$. If $w = 1$, this reduces to the standard definition of Beurling density and, in this case, we just write $\mathcal{D}^+(\Lambda)$ and $\mathcal{D}^-(\Lambda)$.

The following is a useful reinterpretation of finite upper and positive lower weighted density conditions. The first lemma is an extension of [3, Lem. 2.3] to weighted density. Its proof uses the same ideas as the proof in [3], therefore we omit it.

Lemma 2.1. *Let $\Lambda \subset \mathbb{R}^d$ and $w : \Lambda \to \mathbb{R}^+$. Then the following conditions are equivalent.*

(i) $\mathcal{D}^+(\Lambda, w) < \infty$.

(ii) *There exist $h > 0$ and $N < \infty$ such that $\#_w(\Lambda \cap Q_h(x)) < N$ for all $x \in \mathbb{R}^d$.*

(iii) *For each $h > 0$, there exists $N = N(h) < \infty$ such that $\#_w(\Lambda \cap Q_h(x)) < N$ for all $x \in \mathbb{R}^d$.*

A similar result holds for the case of positive lower weighted density. The proof for the non-weighted case can be found in [1].

Lemma 2.2. *Let $\Lambda \subset \mathbb{R}^d$ and $w : \Lambda \to \mathbb{R}^+$. Then the following conditions are equivalent.*

(i) $\mathcal{D}^-(\Lambda, w) > 0$.

(ii) *There exist $h, N > 0$ such that $\#_w(\Lambda \cap Q_h(x)) > N$ for all $x \in \mathbb{R}^d$.*

2.2. **Notation for frames and Bessel sequences.** A system $\{f_i\}_{i\in I}$ in $L^2(\mathbb{R}^d)$ is called a *frame* for $L^2(\mathbb{R}^d)$, if there exist $0 < A \le B < \infty$ (*lower and upper frame bounds*) such that $A\,\|f\|_2^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\,\|f\|_2^2$ for all $f \in L^2(\mathbb{R}^d)$. If A, B can be chosen such that $A = B$, then $\{f_i\}_{i\in I}$ is a *tight frame*, and if we can take $A = B = 1$, it is called a *Parseval frame.* A *Bessel sequence* $\{f_i\}_{i\in I}$ is only required to fulfill the upper frame bound estimate but not necessarily the lower estimate.

2.3. **Amalgam spaces and modulation spaces.** Both amalgam spaces and modulation spaces play an essential role in studying Gabor systems and the short-time Fourier transform. In this paper we employ the following special cases of amalgam and modulation spaces. For more details on amalgam and modulation spaces we refer the reader to [7] and [8].

Given $1 \le p < \infty$, a function $f : \mathbb{R}^d \to \mathbb{C}$ belongs to the *amalgam space* $W(L^\infty, L^p)$ if

$$\|f\|_{W(L^\infty,L^p)} = \left(\sum_{k\in\mathbb{Z}^d} (\text{ess sup}_{x\in Q_1(k)} |f(x)|)^p \right)^{\frac{1}{p}} < \infty.$$

The amalgam space $W(C, L^p)$ is the closed subspace of $W(L^\infty, L^p)$ consisting of the continuous functions in $W(L^\infty, L^p)$.

Let $\gamma(x) = 2^{\frac{d}{4}} e^{-\pi\|x\|^2}$ be the Gaussian function. Then the *modulation space* $M^1(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\|f\|_{M^1} = \|V_\gamma f\|_1 < \infty$. In fact, $M^1(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$.

If $f, g \in L^2(\mathbb{R}^d)$, then $V_g f$ is a continuous L^2-function, and $\|V_g f\|_2 = \|g\|_2\,\|f\|_2$. Further notice that $|V_g f(z)|^2 = |V_f g(-z)|^2$ for each $z \in \mathbb{R}^{2d}$. In the sequel, for $z = (z_1, z_2) \in \mathbb{R}^{2d}$ and $g \in L^2(\mathbb{R}^d)$, we will use the notation $\rho(z)g(x) = e^{2\pi i\langle x, z_2\rangle} g(x - z_1)$, $x \in \mathbb{R}^d$.

The following lemma shows that, if we require that the window g lies in $M^1(\mathbb{R}^d)$, the short-time Fourier transform satisfies additional decay properties.

Lemma 2.3. *Let $g \in M^1(\mathbb{R}^d)$. Then $|V_g f|^2 \in W(C, L^1)$ for each $f \in L^2(\mathbb{R}^d)$.*

Proof. By [7, Prop. 12.1.11], $g \in M^1(\mathbb{R}^d)$ implies $V_g g \in W(L^\infty, L^1)$. Fix $f \in L^2(\mathbb{R}^d)$. Then we have $V_g f * V_g g = \|g\|_2^2\, V_g f$, since

$$(V_g f * V_g g)(z) = \int_{\mathbb{R}^{2d}} V_g f(\tilde{z})\, \overline{V_g(\rho(z)g)(\tilde{z})}\, d(\tilde{z}) = \|g\|_2^2\, \langle f, \rho(z)g\rangle = \|g\|_2^2\, V_g f(z).$$

By [8, Cor. 11.8.4], this implies

$$V_g f = \|g\|_2^{-2}\, (V_g f * V_g g) \in L^2(\mathbb{R}^d) * W(L^\infty, L^1) \subseteq W(L^\infty, L^2).$$

Since $V_g f$ is continuous, the claim follows. $\square$

3. Weighted Beurling density

Considering the definition of Beurling density, we may ask whether we can also measure density using a different shape than cubes. In fact, we will show that we can replace $\chi_{Q_h(0)}$ by an arbitrary piecewise continuous, positive function in $W(L^\infty, L^1)$, yet still derive the same density. We will say that a function $f : \mathbb{R}^d \to \mathbb{C}$ is *piecewise continuous* if there exists a tiling $\{B_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}^d$ of measurable sets with non-empty interiors whose boundaries have measure zero such that $f|_{B_n}$ is continuous for each $n \in \mathbb{N}$.

3.1. Littlewood–Paley type inequalities and weighted Beurling density.

First we study relations between certain Littlewood–Paley type inequalities and weighted Beurling density. Although interesting in their own right, our main purpose is to employ these results in the proof of Theorem 3.5.

We start with some technical lemmas.

Lemma 3.1. *Let $\Lambda \subset \mathbb{R}^d$, $w : \Lambda \to \mathbb{R}^+$, and $f \in W(L^\infty, L^1)$ be piecewise continuous with $f \geq 0$, $f \neq 0$. If there exists $B < \infty$ such that*

$$\sum_{\lambda\in\Lambda} w(\lambda) f(\lambda + x) \leq B \quad \textit{for all } x \in \mathbb{R}^d,$$

then $\mathcal{D}^+(\Lambda, w) < \infty$.

Proof. By hypothesis, since $f \geq 0$ and f is piecewise continuous, there exist some $x_0 \in \mathbb{R}^d$ and $h_0 > 0$ such that $f(x) \geq \delta > 0$ for all $x \in Q_{h_0}(x_0)$. Towards a contradiction assume that $\mathcal{D}^+(\Lambda, w) = \infty$. Applying Lemma 2.1, for each $n \in \mathbb{N}$, there exists some $y_n \in \mathbb{R}^d$ with

$$\#_w(\Lambda \cap Q_{h_0}(y_n)) \geq n.$$

Hence,

$$\sum_{\lambda\in\Lambda} w(\lambda) f(\lambda + (x_0 - y_n)) \geq \sum_{\lambda\in\Lambda\cap Q_{h_0}(y_n)} w(\lambda) f(\lambda + (x_0 - y_n)) \geq \delta\, n,$$

a contradiction. $\square$

Lemma 3.2. *Let $\Lambda \subset \mathbb{R}^d$, $w : \Lambda \to \mathbb{R}^+$, and let $f \in W(L^\infty, L^1)$ be piecewise continuous with $f \geq 0$. If $\mathcal{D}^+(\Lambda, w) < \infty$, then for each $\varepsilon > 0$ there exists $r < \infty$ such that*

$$\sum_{\lambda\in\Lambda} w(\lambda)\, f(\lambda + x)\, \chi_{\mathbb{R}^d\setminus Q_r}(\lambda + x) < \varepsilon \quad \textit{for all } x \in \mathbb{R}^d.$$

Proof. Fix $\varepsilon > 0$. Since $\mathcal{D}^+(\Lambda, w) < \infty$, Lemma 2.1 implies the existence of some $N < \infty$ such that $\#_w(\Lambda \cap Q_1(x)) < N$ for all $x \in \mathbb{R}^d$. Since $f \in W(L^\infty, L^1)$, we can choose $r \in \mathbb{N}$ such that

$$\sum_{k\in\mathbb{Z}^d\setminus Q_{r-1}} \sup_{x\in Q_1(k)} f(x) < \frac{\varepsilon}{N}.$$

Therefore

$$\begin{aligned}\sum_{\lambda\in\Lambda} w(\lambda)\, f(\lambda+x)\, \chi_{\mathbb{R}^d\setminus Q_r}(\lambda+x) &= \sum_{\lambda\in\Lambda\cap(\mathbb{R}^d\setminus Q_r(-x))} w(\lambda)\, f(\lambda+x)\\ &\le \sum_{k\in\mathbb{Z}^d\setminus Q_{r-1}} \sum_{\lambda\in\Lambda\cap Q_1(k-x)} w(\lambda)\, f(\lambda+x)\\ &< N \sum_{k\in\mathbb{Z}^d\setminus Q_{r-1}} \sup_{y\in Q_1(k)} f(y)\\ &< \varepsilon. \end{aligned}$$ □

We can now state and prove the main result of this subsection.

Theorem 3.3. *Let $\Lambda \subset \mathbb{R}^d$, $w : \Lambda \to \mathbb{R}^d$, and let $f \in W(L^\infty, L^1)$ be piecewise continuous with $f \ge 0$. If*

$$A \le \sum_{\lambda\in\Lambda} w(\lambda)\, f(\lambda+x) \le B \quad \textit{for all } x \in \mathbb{R}^d, \tag{3}$$

then

$$A \le \mathcal{D}^-(\Lambda, w)\, \|f\|_1 \le \mathcal{D}^+(\Lambda, w)\, \|f\|_1 \le B.$$

Proof. We first claim that for each $\varepsilon > 0$ there exists $r < \infty$ with

$$A - \varepsilon \le \sum_{\lambda\in\Lambda} w(\lambda)\, f(\lambda+x)\, \chi_{Q_r}(\lambda+x) \le B \quad \text{for all } x \in \mathbb{R}^d. \tag{4}$$

To prove this, fix some $\varepsilon > 0$. Noting that $\mathcal{D}^+(\Lambda, w) < \infty$ by Lemma 3.1, the application of Lemma 3.2 implies that there exists $r < \infty$ with

$$\sum_{\lambda\in\Lambda} w(\lambda)\, f(\lambda+x)\, \chi_{\mathbb{R}^d\setminus Q_r}(\lambda+x) < \varepsilon \quad \text{for all } x \in \mathbb{R}^d.$$

Combining this with (3) proves (4).

Let $\varepsilon > 0$. Without loss of generality we can assume that r is large enough so that $\|f\|_1 - \int_{Q_r} f(x)\, dx < \varepsilon$. Fix $y \in \mathbb{R}^d$ and $h > r$. Integrating each term in (4) over the box $Q_h(-y)$ yields

$$(A - \varepsilon) h^d \le \sum_{\lambda\in\Lambda} w(\lambda) \int_{Q_h(-y)} f(\lambda+x)\, \chi_{Q_r}(\lambda+x)\, dx \le B h^d. \tag{5}$$

Then we make the decomposition

$$\sum_{\lambda\in\Lambda} w(\lambda) \int_{Q_h(-y)} f(\lambda+x)\, \chi_{Q_r}(\lambda+x)\, dx = I_1(y,h) - I_2(y,h) + I_3(y,h) + I_4(y,h),$$

where

$$\begin{aligned}
I_1(y,h) &= \sum_{\lambda\in\Lambda\cap Q_{h-r}(y)} w(\lambda)\int_{\mathbb{R}^d} f(\lambda+x)\,\chi_{Q_r}(\lambda+x)\,dx,\\
I_2(y,h) &= \sum_{\lambda\in\Lambda\cap Q_{h-r}(y)} w(\lambda)\int_{\mathbb{R}^d\setminus Q_h(-y)} f(\lambda+x)\,\chi_{Q_r}(\lambda+x)\,dx,\\
I_3(y,h) &= \sum_{\lambda\in\Lambda\cap(Q_{h+r}(y)\setminus Q_{h-r}(y))} w(\lambda)\int_{Q_h(-y)} f(\lambda+x)\,\chi_{Q_r}(\lambda+x)\,dx,\\
I_4(y,h) &= \sum_{\lambda\in\Lambda\cap(\mathbb{R}^d\setminus Q_{h+r}(y))} w(\lambda)\int_{Q_h(-y)} f(\lambda+x)\,\chi_{Q_r}(\lambda+x)\,dx.
\end{aligned}$$

Since $\mathcal{D}^+(\Lambda,w)<\infty$, by Lemma 2.1 there exists $N<\infty$ such that for each $s>0$ we have $\#_w(\Lambda\cap Q_s(x_0))\le(s+1)^d\sup_{x\in\mathbb{R}^d}\#_w(\Lambda\cap Q_1(x))<(s+1)^dN$ for all $x_0\in\mathbb{R}^d$.

We first observe that

$$I_1(y,h)=\sum_{\lambda\in\Lambda\cap Q_{h-r}(y)} w(\lambda)\int_{Q_r} f(x)\,dx=\#_w(\Lambda\cap Q_{h-r}(y))\int_{Q_r} f(x)\,dx,$$

and hence, by the choice of r,

$$\left|I_1(y,h)-\#_w(\Lambda\cap Q_{h-r}(y))\,\|f\|_1\right|<\#_w(\Lambda\cap Q_{h-r}(y))\,\varepsilon.$$

The contribution of $I_2(y,h)$ to the sum in (5) is

$$I_2(y,h)=\sum_{\lambda\in\Lambda\cap Q_{h-r}(y)} w(\lambda)\int_{(\mathbb{R}^d\setminus Q_h(\lambda-y))\cap Q_r} f(x)\,dx=0,$$

since $\|\lambda-y\|_\infty\le\frac{h-r}{2}$, and hence $(\mathbb{R}^d\setminus Q_h(\lambda-y))\cap Q_r$ has Lebesgue measure zero. The term $I_3(y,h)$ can be estimated as follows:

$$\begin{aligned}
I_3(y,h) &= \sum_{\lambda\in\Lambda\cap(Q_{h+r}(y)\setminus Q_{h-r}(y))} w(\lambda)\int_{Q_h(-y)\cap Q_r(-\lambda)} f(\lambda+x)\,dx\\
&\le \#_w(\Lambda\cap(Q_{h+r}(y)\setminus Q_{h-r}(y)))\,\|f\|_1\\
&< ((h+r+1)^d-(h-r+1)^d)N\,\|f\|_1\,.
\end{aligned}$$

Finally, the contribution of $I_4(y,h)$ is

$$I_4(y,h)=\sum_{\lambda\in\Lambda\cap(\mathbb{R}^d\setminus Q_{h+r}(y))} w(\lambda)\int_{Q_h(\lambda-y)\cap Q_r} f(x)\,dx=0,$$

since $\|\lambda-y\|_\infty\ge\frac{h+r}{2}$, and hence $Q_h(\lambda-y)\cap Q_r$ has Lebesgue measure zero.

Combining the above estimates, we see that

$$(A-\varepsilon)h^d\le\#_w(\Lambda\cap Q_{h-r}(y))(\|f\|_1+\varepsilon)+\big((h+r+1)^d-(h-r+1)^d\big)N\,\|f\|_1\quad\text{for all } y\in\mathbb{R}^d.$$

Therefore

$$\begin{aligned}
&A-\varepsilon\\
&= \liminf_{h\to\infty} \frac{(A-\varepsilon)h^d}{h^d}\\
&\le \liminf_{h\to\infty} \inf_{y\in\mathbb{R}^d} \frac{\#_w(\Lambda\cap Q_{h-r}(y))(\|f\|_1+\varepsilon)}{h^d} + \limsup_{h\to\infty} \frac{((h+r+1)^d-(h-r+1)^d)N\|f\|_1}{h^d}\\
&= \mathcal{D}^-(\Lambda,w)(\|f\|_1+\varepsilon).
\end{aligned}$$

Now letting ε go to zero yields $A \le \mathcal{D}^-(\Lambda,w)\|f\|_1$. The second claim $\mathcal{D}^+(\Lambda,w)\|f\|_1 \le B$ can be treated similarly. The theorem is proved. $\square$

3.2. **Equivalent definition of weighted Beurling density.** The precise definition of the general notion of density with respect to an arbitrary piecewise continuous, positive function in $W(L^\infty, L^1)$ is as follows. As usual, we define the dilation operator D_h on $L^1(\mathbb{R}^d)$ by $D_h f(x) = \frac{1}{h^d} f(\frac{x}{h})$.

Definition 3.4. Let $f \in W(L^\infty, L^1)$ be piecewise continuous with $f \ge 0$, $f \ne 0$, and let $\Lambda \subset \mathbb{R}^d$, $w : \Lambda \to \mathbb{R}^+$ be given. Then the *upper weighted f-density of the pair* (Λ, w) is

$$\mathcal{D}_f^+(\Lambda,w) = \|f\|_1^{-1} \limsup_{h\to\infty} \sup_{x\in\mathbb{R}^d} \sum_{\lambda\in\Lambda} w(\lambda) D_h f(\lambda+x)$$

and the *lower weighted f-density of* (Λ, w) is

$$\mathcal{D}_f^-(\Lambda,w) = \|f\|_1^{-1} \liminf_{h\to\infty} \inf_{x\in\mathbb{R}^d} \sum_{\lambda\in\Lambda} w(\lambda) D_h f(\lambda+x).$$

The Beurling density is indeed a special case of this definition, since for $\Lambda \subset \mathbb{R}^d$ and $w : \Lambda \to \mathbb{R}^+$, the lower weighted Beurling density $\mathcal{D}^-(\Lambda, w)$ coincides with $\mathcal{D}^-_{\chi_{[0,1)^d}}(\Lambda, w)$ and the upper density $\mathcal{D}^+(\Lambda, w)$ coincides with $\mathcal{D}^+_{\chi_{[0,1)^d}}(\Lambda, w)$.

The following result now shows that in fact both definitions coincide for any f. In particular, $\mathcal{D}_f^+(\Lambda, w)$ and $\mathcal{D}_f^-(\Lambda, w)$ do not depend on the function f. This result contains [12, Lem. 4] as a special case.

Theorem 3.5. *Let $f \in W(L^\infty, L^1)$ be piecewise continuous with $f \ge 0$, $f \ne 0$. Then, for all $\Lambda \subset \mathbb{R}^d$ and $w : \Lambda \to \mathbb{R}^+$, we have*

$$\mathcal{D}_f^-(\Lambda,w) = \mathcal{D}^-(\Lambda,w) \quad \textit{and} \quad \mathcal{D}_f^+(\Lambda,w) = \mathcal{D}^+(\Lambda,w).$$

Proof. We will only prove $\mathcal{D}_f^+(\Lambda, w) = \mathcal{D}^+(\Lambda, w)$. The other claim can be treated in a similar manner.

First we observe that, without loss of generality, we can assume that $\|f\|_1 = 1$ by rescaling. Now, towards a contradiction, assume that $\mathcal{D}_f^+(\Lambda, w) < \mathcal{D}^+(\Lambda, w)$. Then there exist $\delta > 0$ and $\tilde{h} > 0$ such that for all $h \ge \tilde{h}$ and $x \in \mathbb{R}^d$,

$$\sum_{\lambda\in\Lambda} w(\lambda) D_h f(\lambda+x) \le \mathcal{D}^+(\Lambda,w) - \delta.$$

This in turn implies that there exist $\varepsilon \in (0, \delta)$, $(h_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ with $h_n \to \infty$ as $n \to \infty$, and $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ such that for all $h \geq \tilde{h}$, $n \in \mathbb{N}$, and $x \in \mathbb{R}^d$,

$$\sum_{\lambda\in\Lambda} w(\lambda) D_h f(\lambda + x) \leq \frac{\#_w(\Lambda \cap Q_{h_n}(x_n))}{h_n^d} - \varepsilon. \tag{6}$$

Without loss of generality we can choose $(h_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ so that

$$\left| \frac{\#_w(\Lambda \cap Q_{h_n}(x_n))}{h_n^d} - \mathcal{D}^+(\Lambda, w) \right| < \frac{\varepsilon}{2} \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

Since $f \in W(L^\infty, L^1)$, we also have $D_h f \in W(L^\infty, L^1)$ for all $h > 0$. Thus we can apply Theorem 3.3 to (6), which yields

$$\mathcal{D}^+(\Lambda, w) \, \|D_h f\|_1 \leq \frac{\#_w(\Lambda \cap Q_{h_n}(x_n))}{h_n^d} - \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

By (7) and using that $\|D_h f\|_1 = 1$,

$$\mathcal{D}^+(\Lambda, w) \leq \mathcal{D}^+(\Lambda, w) - \frac{\varepsilon}{2},$$

a contradiction. Thus $\mathcal{D}_f^+(\Lambda, w) \geq \mathcal{D}^+(\Lambda, w)$.

To prove $\mathcal{D}_f^+(\Lambda, w) \leq \mathcal{D}^+(\Lambda, w)$, we first study this claim for a certain step function. Let $\nu > 0$, set $c_k = \sup_{x\in Q_\nu(\nu k)} f(x)$ for each $k \in \mathbb{Z}^d$, and define $g : \mathbb{R}^d \to \mathbb{R}^+$ by

$$g = \sum_{k\in\mathbb{Z}^d} c_k \chi_{Q_\nu(\nu k)}.$$

Let $\varepsilon > 0$. Since $f \in W(L^\infty, L^1)$, we have $\|g\|_1 < \infty$ regardless of the value of ν. Choose ν so that $\|g - f\|_1 < \varepsilon$. Let $K \subset \mathbb{R}^d$ be compact. Then there exists a compact $\tilde{K} \subseteq \mathbb{R}^d$ such that

$$(g\chi_K)(x) \leq \sum_{k\in\mathbb{Z}^d\cap\tilde{K}} c_k \chi_{Q_\nu(\nu k)}(x) \quad \text{for all } x \in \mathbb{R}^d.$$

Since $\|f\|_1 = 1$, we further have

$$\nu^d \sum_{k\in\mathbb{Z}^d\cap\tilde{K}} c_k \leq \|g\|_1 \leq 1 + \varepsilon.$$

Collecting these arguments, for $h > 0$, we obtain

$$\begin{aligned}
\sup_{x\in\mathbb{R}^d} \sum_{\lambda\in\Lambda} w(\lambda) D_h(g\chi_K)(\lambda + x) &\leq \sup_{x\in\mathbb{R}^d} \frac{\sum_{\lambda\in\Lambda} w(\lambda) \sum_{k\in\mathbb{Z}^d\cap\tilde{K}} c_k \chi_{Q_\nu(\nu k)}(\frac{\lambda+x}{h})}{h^d} \\
&= \sup_{x\in\mathbb{R}^d} \frac{\sum_{k\in\mathbb{Z}^d\cap\tilde{K}} c_k \sum_{\lambda\in\Lambda} w(\lambda) \chi_{hQ_\nu(\nu k)-x}(\lambda)}{h^d} \\
&\leq \sum_{k\in\mathbb{Z}^d\cap\tilde{K}} c_k \sup_{x\in\mathbb{R}^d} \frac{\sum_{\lambda\in\Lambda} w(\lambda) \chi_{Q_{h\nu}(h\nu k - x)}(\lambda)}{h^d}
\end{aligned}$$

$$
\begin{aligned}
&= \nu^d \sum_{k\in\mathbb{Z}^d\cap\tilde{K}} c_k \sup_{x\in\mathbb{R}^d} \frac{\#_w(\Lambda\cap Q_{h\nu}(x))}{h^d\nu^d} \\
&\leq (1+\varepsilon)\sup_{x\in\mathbb{R}^d} \frac{\#_w(\Lambda\cap Q_{h\nu}(x))}{h^d\nu^d}. \qquad (8)
\end{aligned}
$$

We assume that $\mathcal{D}^+(\Lambda,w)<\infty$, since otherwise the claim is already proved. By Lemma 3.2, there exists some $r_h<\infty$ with

$$\sum_{\lambda\in\Lambda} w(\lambda)\, D_h(f\chi_{\mathbb{R}^d\setminus Q_{r_h}})(\lambda+x) < \varepsilon \quad \text{for all } x\in\mathbb{R}^d,$$

which implies

$$\left|\sum_{\lambda\in\Lambda} w(\lambda)\, D_h f(\lambda+x) - \sum_{\lambda\in\Lambda} w(\lambda)\, D_h(f\chi_{Q_{r_h}})(\lambda+x)\right| < \varepsilon \quad \text{for all } x\in\mathbb{R}^d. \qquad (9)$$

Using (9), the fact that $f\leq g$, and (8), we obtain

$$
\begin{aligned}
\sup_{x\in\mathbb{R}^d}\sum_{\lambda\in\Lambda} w(\lambda)\, D_h f(\lambda+x) &\leq \sup_{x\in\mathbb{R}^d}\sum_{\lambda\in\Lambda} w(\lambda)\, D_h(f\chi_{Q_{r_h}})(\lambda+x)+\varepsilon \\
&\leq \sup_{x\in\mathbb{R}^d}\sum_{\lambda\in\Lambda} w(\lambda)\, D_h(g\chi_{Q_{r_h}})(\lambda+x)+\varepsilon \\
&\leq (1+\varepsilon)\sup_{x\in\mathbb{R}^d} \frac{\#_w(\Lambda\cap Q_{h\nu}(x))}{h^d\nu^d}+\varepsilon.
\end{aligned}
$$

Applying the limsup as $h\to\infty$ yields

$$\mathcal{D}_f^+(\Lambda,w)\leq(1+\varepsilon)\mathcal{D}^+(\Lambda,w)+\varepsilon.$$

Letting ε go to zero therefore finishes the proof. □

Provided that $g\in M^1(\mathbb{R}^d)$, we might use the associated short-time Fourier transform to measure density.

Corollary 3.6. *Let $g\in M^1(\mathbb{R}^d)$. Then, for all $f\in L^2(\mathbb{R}^d)$, $\Lambda\subset\mathbb{R}^{2d}$, and $w:\Lambda\to\mathbb{R}^+$, we have*

$$\mathcal{D}^-_{|V_gf|^2}(\Lambda,w)=\mathcal{D}^-(\Lambda,w) \quad \textit{and} \quad \mathcal{D}^+_{|V_gf|^2}(\Lambda,w)=\mathcal{D}^+(\Lambda,w).$$

Proof. This follows immediately from Lemma 2.3 and Theorem 3.5. □

If we are only interested in estimates and are equipped with an upper and lower bound for $\sum_{\lambda\in\Lambda} w(\lambda)D_1f(\lambda+x)$ uniformly in $x\in\mathbb{R}^d$, this immediately yields estimates for $\mathcal{D}_f^-(\Lambda,w)$ and $\mathcal{D}_f^+(\Lambda,w)$.

Corollary 3.7. *Let $f\in W(L^\infty,L^1)$ be piecewise continuous with $f\geq 0$, and let $\Lambda\subset\mathbb{R}^d$, $w:\Lambda\to\mathbb{R}^+$. If*

$$A\leq\sum_{\lambda\in\Lambda} w(\lambda)\, f(\lambda+x)\leq B \quad \textit{for all } x\in\mathbb{R}^d,$$

then

$$A \le \mathcal{D}_f^-(\Lambda, w)\, \|f\|_1 \le \mathcal{D}_f^+(\Lambda, w)\, \|f\|_1 \le B.$$

Proof. Applying Theorem 3.3 and Theorem 3.5 yields the claim. □

4. Shift-invariant weighted irregular Gabor systems

Turning a Gabor frame into a shift-invariant weighted Gabor frame imposes conditions not only on the set of indices of the original frame, but also on the relation between the densities of both sets of indices. These issues will be discussed in the following subsections. Moreover, we will study whether the necessary condition we derive can be fulfilled, i.e., whether appropriate weight functions can be constructed.

4.1. **A fundamental relationship for weighted Gabor systems.** Provided that a weighted Gabor system forms a frame, we will derive a fundamental relationship between the weighted Beurling density of its set of indices, the frame bounds, and the norm of the generator.

First, however, we will require the following technical lemma. Provided that a Gabor system forms a frame, it shows that the associated short-time Fourier transform satisfies a certain inequality of Littlewood–Paley type. The unweighted case is well-known.

Lemma 4.1. *Let $\Lambda \subset \mathbb{R}^{2d}$ and $w : \Lambda \to \mathbb{R}^+$. Let $g \in L^2(\mathbb{R}^d)$ be such that $\mathcal{G}(g, \Lambda, w)$ is a frame for $L^2(\mathbb{R}^d)$ with frame bounds A and B. Then for all $f \in L^2(\mathbb{R}^d)$ with $\|f\|_2 = 1$, we have*

$$A \le \sum_{\lambda \in \Lambda} w(\lambda)\, |V_g f(\lambda + z)|^2 \le B \quad \text{for all } z \in \mathbb{R}^{2d}.$$

Proof. Since $\mathcal{G}(g, \Lambda, w)$ is a frame,

$$A\, \|f_0\|_2^2 \le \sum_{\lambda \in \Lambda} w(\lambda)\, |\langle f_0, \rho(\lambda) g\rangle|^2 \le B\, \|f_0\|_2^2 \quad \text{for all } f_0 \in L^2(\mathbb{R}^d).$$

Fix $f \in L^2(\mathbb{R}^d)$ with $\|f\|_2 = 1$. Choosing $f_0 = \rho(-z) f$, where $z \in \mathbb{R}^{2d}$ is arbitrary, yields

$$A \le \sum_{\lambda \in \Lambda} w(\lambda)\, |\langle \rho(-z) f, \rho(\lambda) g\rangle|^2 = \sum_{\lambda \in \Lambda} w(\lambda)\, |\langle f, \rho(\lambda + z) g\rangle|^2 \le B.$$

Using the definition of $V_g f$ finishes the proof. □

Notice that, provided $g \in M^1$, the tails of the sum considered in Lemma 4.1 are arbitrarily small due to Lemmas 2.3 and 3.2.

The fundamental relationship between the weighted Beurling density of the set of indices, the frame bounds, and the norm of the generator of weighted Gabor frames now follows from all our previous discussions.

Theorem 4.2. *Let $g \in L^2(\mathbb{R}^d)$, $\Lambda \subset \mathbb{R}^{2d}$, and $w : \Lambda \to \mathbb{R}^+$. If $\mathcal{G}(g, \Lambda, w)$ is a frame for $L^2(\mathbb{R}^d)$ with frame bounds A and B, then*

$$A \le \mathcal{D}^-(\Lambda, w)\, \|g\|_2^2 \le \mathcal{D}^+(\Lambda, w)\, \|g\|_2^2 \le B.$$

Proof. Fix $f \in M^1(\mathbb{R}^d)$ with $\|f\|_2 = 1$, and consider $|V_g f|^2$. Applying Lemma 4.1, Lemma 2.3, Corollary 3.7, and Corollary 3.6 to this function, and using the facts that $|V_g f(z)|^2 = |V_f g(-z)|^2$ for $z \in \mathbb{R}^{2d}$ and that $\| |V_g f|^2 \|_1 = \|f\|_2^2 \, \|g\|_2^2$, then settles the claim. $\square$

The previous theorem implies that a weighted Gabor system can only form a tight frame when the weighted density of its set of indices is uniform.

Corollary 4.3. *Let $g \in L^2(\mathbb{R}^d)$, $\Lambda \subset \mathbb{R}^{2d}$, and $w : \Lambda \to \mathbb{R}^+$ be such that $\mathcal{G}(g, \Lambda, w)$ is a tight frame for $L^2(\mathbb{R}^d)$ with frame bound A. Then (Λ, w) has uniform weighted density, and*

$$A = \mathcal{D}(\Lambda, w) \, \|g\|_2^2 \, .$$

In the following remark we point out the difference between the situation of weighted and non-weighted Gabor frames.

Remark 4.4. Let $g \in L^2(\mathbb{R}^d)$, $\Lambda \subset \mathbb{R}^{2d}$, and $w : \Lambda \to \mathbb{R}^+$ be such that $\mathcal{G}(g, \Lambda, w)$ is a frame for $L^2(\mathbb{R}^d)$. First we observe that this forces the weights to be bounded from above. To prove this, towards a contradiction assume that there exists a sequence $(\lambda_n)_{n \in \mathbb{N}} \subseteq \Lambda$ such that $w(\lambda_n) \geq n$ for all $n \in \mathbb{N}$. Then $\#_w(\Lambda \cap Q_h(\lambda_n)) \geq n$, and hence $\sup_{x \in \mathbb{R}^d} \#_w(\Lambda \cap Q_h(x)) = \infty$. It follows that $\mathcal{D}^+(\Lambda, w) = \infty$. But, by Theorem 4.2, this implies that $\mathcal{G}(g, \Lambda, w)$ does not form a frame for $L^2(\mathbb{R}^d)$, which is a contradiction. Thus w is bounded from above.

However, w need not necessarily be bounded from below. This can easily be seen by starting with a frame $\mathcal{G}(g, \Lambda, w)$, choosing one element from Λ, and then adding infinitely many repetitions of this element to Λ, with corresponding weights that decrease to zero. If the weights decrease quickly enough, then the new system will still be a frame for $L^2(\mathbb{R}^d)$. By perturbing the new points, less trivial examples can also be generated. Thus we can construct weighted Gabor frames with arbitrarily small weights.

It seems natural to ask whether each weighted Gabor frame contains a subset that is a non-weighted (irregular) Gabor frame. Considering the procedure of repeating one element of the set of indices, it is obvious that deleting those frame elements with small weights below a threshold will lead to a non-complete Gabor system in most cases. However, there might exist more sophisticated ways to delete certain elements from a weighted Gabor frame. We suspect that this can be done.

In this context there exists a very interesting approach towards non-uniform sampling theory by employing weights due to Feichtinger, Gröchenig, and Strohmer [6]. They introduced the so-called "adaptive weights method" to compensate for local variations of the set of sampling points, thereby deriving "superfast" algorithms for the reconstruction from these points. This should be compared to the fact that the introduction of weights for the study of shift-invariant Gabor frames can be seen as a consequence of enlarging the set of indices, which the added weights have to compensate for.

4.2. **Existence of an associated discrete shift-invariant set.** Given a Gabor system $\mathcal{G}(g, \Lambda)$, we are interested in the existence and properties of the smallest discrete set $\tilde{\Lambda} \subset \mathbb{R}^{2d}$ containing Λ so that $\mathcal{G}(g, \tilde{\Lambda})$ is *shift-invariant*, i.e., $T_k \mathcal{G}(g, \tilde{\Lambda}) \subseteq \mathcal{G}(g, \tilde{\Lambda})$ for all $k \in \mathbb{Z}^d$, where $T_k : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ denotes the shift operator $T_k f(x) = f(x - k)$. As usual we call a subset $\tilde{\Lambda}$ of $\mathbb{R}^{2d}$ *discrete*, if for each $\lambda \in \tilde{\Lambda}$ there exists an open set $U \subset \mathbb{R}^{2d}$ with $U \cap \tilde{\Lambda} = \{\lambda\}$.

Our first result shows that we indeed have an explicit representation of the set under consideration.

Proposition 4.5. *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^{2d}$. Further let $\Delta \subset \mathbb{R}^{2d}$ be such that $\mathcal{G}(g, \Delta)$ is a shift-invariant system. If $\Lambda \subseteq \Delta$, then $\bigcup_{k \in \mathbb{Z}^d} \Lambda + (k, 0) \subseteq \Delta$.*

Proof. Fix $(a, b) \in \mathbb{R}^{2d}$ and $k \in \mathbb{Z}$. Then

$$T_k(\rho(a, b)g)(x) = e^{2\pi ibx} g(x - (k + a)) = (\rho(a + k, b)g)(x) = (\rho((a, b) + (k, 0))g)(x).$$

Thus, provided that $\Lambda \subseteq \Delta$ and $\mathcal{G}(g, \Delta)$ is a shift-invariant system, then $(a, b) + (k, 0) \in \Delta$ for all $(a, b) \in \Lambda$ and $k \in \mathbb{Z}^d$. This proves the lemma. □

This shows that, given a subset Λ of $\mathbb{R}^{2d}$, there exists a canonical larger subset, which provides us with a shift-invariant Gabor system.

Definition 4.6. Let Λ be a subset of $\mathbb{R}^{2d}$. Then we define the *associated shift-invariant set* Λ_{SI} by $\Lambda_{SI} = \bigcup_{k \in \mathbb{Z}^d} \Lambda + (k, 0)$.

Since we are dealing with discrete Gabor systems, we are only interested in situations when the associated shift-invariant set is also discrete. In fact, we derive an exact characterization of those sets Λ.

Proposition 4.7. *Let $\Lambda \subset \mathbb{R}^{2d}$ be discrete. Then the following conditions are equivalent.*

(i) *Λ_{SI} is discrete.*

(ii) *There exists a discrete set $T \subset \mathbb{R}^d$ such that for each $t \in T$ there exist $s_1^t, \ldots, s_{n_t}^t \in [0, 1)^d$, $n_t \in \mathbb{N}$ such that $\Lambda \subseteq \bigcup_{t \in T} \left(\bigcup_{i=1}^{n_t} s_i^t + \mathbb{Z}^d\right) \times \{t\}$.*

Proof. In the following, abusing notation we write $x \bmod 1$, $x \in \mathbb{R}^d$, for the modulus taken in each component, hence $x \bmod 1 \in [0, 1)^d$.

First we prove (i) $\Rightarrow$ (ii). Let P_2 denote the orthogonal projection of $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$ onto the second component. Towards a contradiction, assume that there exists $t_0 \in P_2(\Lambda)$ such that $\#\{s \bmod 1 : (s, t_0) \in \Lambda\} = \infty$. Then $(\bigcup_{(s,t_0) \in \Lambda} s + \mathbb{Z}^d) \cap [0, 1)^d$ would contain infinitely many elements, hence would not be discrete, in contradiction to (i). Thus $\#\{s \bmod 1 : (s, t_0) \in \Lambda\} < \infty$ for each $t_0 \in P_2(\Lambda)$. Again towards a contradiction assume that $P_2(\Lambda)$ is not discrete, i.e., that there exists a convergent sequence $(t_n)_{n \in \mathbb{N}} \subset P_2(\Lambda)$ with limit $t_0 \in P_2(\Lambda)$. Let $(s_n)_{n \in \mathbb{N}_0}$ be such that $(s_n, t_n) \in \Lambda$ for all $n \in \mathbb{N}_0$. Hence $\{(s_n \bmod 1, t_n) : n \in \mathbb{N}_0\} \subseteq \Lambda_{SI}$. But $s_n \bmod 1 \in [0, 1)^d$ for all $n \in \mathbb{N}_0$, thus $(s_n)_{n \in \mathbb{N}}$ contains a convergent subsequence $(s_{n_k})_{k \in \mathbb{N}}$. We obtain that $(s_{n_k}, t_{n_k})_{k \in \mathbb{N}}$ and its limit are contained in Λ_{SI}, in contradiction to (i). Collecting the above two properties of the set Λ yields (ii).

To prove the converse implication suppose that (ii) is satisfied. It is easy to see that $\Lambda_{SI} \subseteq \bigcup_{t \in T} \left(\bigcup_{i=1}^{n_t} s_i^t + \mathbb{Z}^d\right) \times \{t\}$. Thus Λ_{SI} is discrete. □

The following corollary is an immediate consequence of the previous result.

Corollary 4.8. *Let $\Lambda = S \times T \subset \mathbb{R}^{2d}$ be discrete. Then the following conditions are equivalent.*

(i) *Λ_{SI} is discrete.*

(ii) *There exist $s_1, \ldots, s_n \in [0, 1)^d$, $n \in \mathbb{N}$ such that $S \subseteq \bigcup_{i=1}^{n} s_i + \mathbb{Z}^d$.*

A large class of examples consists of arbitrary subsets Λ of lattices $a\mathbb{Z}^d \times b\mathbb{Z}^d$, $a \in \mathbb{Q}^+$, $b > 0$, in which case Λ^{SI} is always discrete.

4.3. **Density conditions.** Extending a Gabor system to a weighted shift-invariant Gabor system while preserving special properties of the system will be shown to impose special conditions on the density of those systems.

Capturing the spirit of the relation between affine and quasi-affine systems and more generally the oversampling theorems in wavelet theory we are led to the following definition in the situation of Gabor systems.

Definition 4.9. Let $\Lambda \subset \mathbb{R}^{2d}$ and $w : \Lambda_{SI} \to \mathbb{R}^+$ be such that there exists $g \in L^2(\mathbb{R}^d)$ so that both $\mathcal{G}(g, \Lambda)$ and $\mathcal{G}(g, \Lambda_{SI}, w)$ are frames for $L^2(\mathbb{R}^d)$ with a common lower frame bound A and common upper frame bound B. Furthermore, suppose that, for each $h \in L^2(\mathbb{R}^d)$, if $\mathcal{G}(h, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ with frame bounds A and B, then $\mathcal{G}(h, \Lambda_{SI}, w)$ is also a frame for $L^2(\mathbb{R}^d)$ with frame bounds A and B. Then we call $(\Lambda, \Lambda_{SI}, w, g)$ a *shift-invariant extension pair with respect to the frame bounds A and B*. We call $(\Lambda, \Lambda_{SI}, w, g)$ a *shift-invariant extension pair with respect to the frame bound A*, if $A = B$.

Interestingly, already the existence of the one function g yields a relation between the densities of Λ and (Λ_{SI}, w), the frame bounds, and the norm of g.

Proposition 4.10. *Let $\Lambda \subset \mathbb{R}^{2d}$ and $w : \Lambda_{SI} \to \mathbb{R}^+$ be such that there exists $g \in L^2(\mathbb{R}^d)$ so that both $\mathcal{G}(g, \Lambda)$ and $\mathcal{G}(g, \Lambda_{SI}, w)$ are frames for $L^2(\mathbb{R}^d)$ with a common lower frame bound A and common upper frame bound B. Then*

$$A \|g\|_2^{-2} \le \mathcal{D}^-(\Lambda) \le \mathcal{D}^+(\Lambda) \le B \|g\|_2^{-2}$$

and

$$A \|g\|_2^{-2} \le \mathcal{D}^-(\Lambda_{SI}, w) \le \mathcal{D}^+(\Lambda_{SI}, w) \le B \|g\|_2^{-2}.$$

Proof. This follows immediately from Theorem 4.2. □

This leads to a necessary condition on the density of Λ and (Λ_{SI}, w) for the existence of a shift-invariant extension pair.

Theorem 4.11. *Let $g \in L^2(\mathbb{R}^d)$, $\Lambda \subset \mathbb{R}^{2d}$, and $w\colon \Lambda_{SI} \to \mathbb{R}^+$ be given. If $(\Lambda, \Lambda_{SI}, w, g)$ is a shift-invariant extension pair with respect to the frame bounds A and B, then*

$$A \|g\|_2^{-2} \le \mathcal{D}^-(\Lambda) \le \mathcal{D}^+(\Lambda) \le B \|g\|_2^{-2}$$

and

$$A \|g\|_2^{-2} \le \mathcal{D}^-(\Lambda_{SI}, w) \le \mathcal{D}^+(\Lambda_{SI}, w) \le B \|g\|_2^{-2}.$$

Proof. Employing the definition of a shift-invariant extension pair and Proposition 4.10 yields the claim. □

Now we consider the special situation in which we have a shift-invariant extension pair with respect to the frame bound A.

Corollary 4.12. *Let $g \in L^2(\mathbb{R}^d)$, $\Lambda \subset \mathbb{R}^{2d}$, and $w : \Lambda_{SI} \to \mathbb{R}^+$. If $(\Lambda, \Lambda_{SI}, w, g)$ is a shift-invariant extension pair with respect to the frame bound A, then both Λ and (Λ_{SI}, w) have uniform density and*

$$\mathcal{D}(\Lambda) = \mathcal{D}(\Lambda_{SI}, w) = A \|g\|_2^{-2}.$$

Proof. This is an immediate consequence of Theorem 4.11. □

Further, we discuss whether for an existing Gabor system, the shift-invariant system can be equipped with weights so that the densities of both sets of indices coincide, a necessary condition for the existence of shift-invariant extension pairs with respect to the frame bound A.

Proposition 4.13. *Let $\Lambda \subset \mathbb{R}^{2d}$ with $0 < \mathcal{D}^-(\Lambda) \leq \mathcal{D}^+(\Lambda) < \infty$. Then there exists a weight function $w : \Lambda_{SI} \to \mathbb{R}^+$ such that*

$$\mathcal{D}^-(\Lambda_{SI}, w) = \mathcal{D}^-(\Lambda) \quad \text{and} \quad \mathcal{D}^+(\Lambda_{SI}, w) = \mathcal{D}^+(\Lambda).$$

Proof. By Lemma 2.2, there exists some minimal $h > 0$ with $\inf_{x\in\mathbb{R}^{2d}} \#(\Lambda \cap Q_h(x)) > 0$. For each $k = (k_1, \ldots, k_{2d}) \in \mathbb{Z}^{2d}$, let $y_k \in \mathbb{R}^{2d}$ be defined by $y_k = (k_1 h, \ldots, k_{2d} h)$. Then define $w : \Lambda \to \mathbb{R}^+$ by

$$w(a) = \frac{\#(\Lambda \cap Q_h(y_k))}{\#(\Lambda_{SI} \cap Q_h(y_k))} \quad \text{for all } a \in \Lambda \cap Q_h(y_k),\ k \in \mathbb{Z}^{2d}.$$

Since w is constructed in such a way that $\#_w(\Lambda_{SI} \cap Q_h(y_k)) = \#(\Lambda \cap Q_h(y_k))$ for all $k \in \mathbb{Z}^{2d}$, for each $x \in \mathbb{R}^{2d}$ and $r > 1$ there exists $k \in \mathbb{Z}^{2d}$ with

$$\#(\Lambda \cap Q_{(r-2)h}(y_k)) \leq \#_w(\Lambda_{SI} \cap Q_{rh}(x)) \leq \#(\Lambda \cap Q_{(r+2)h}(y_k))$$

and

$$\#_w(\Lambda_{SI} \cap Q_{(r-2)h}(y_k)) \leq \#(\Lambda \cap Q_{rh}(x)) \leq \#_w(\Lambda_{SI} \cap Q_{(r+2)h}(y_k)).$$

Employing the definition of weighted density settles the claim. □

At last, we present a short example of a shift-invariant extension pair in the one-dimensional situation to enlighten the previous results.

Example 4.14. Consider the subset $\Lambda = 2\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ of $\mathbb{R}^2$, whose uniform density equals 1. The associated shift-invariant set is easily seen to be $\Lambda_{SI} = \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$. Employing the construction contained in the proof of Proposition 4.13 yields the weight function $w : \Lambda_{SI} \to \mathbb{R}^+$ defined by $w \equiv \frac{1}{2}$. Then $\mathcal{D}(\Lambda) = 1 = \mathcal{D}(\Lambda_{SI}, w)$.

Now let $g \in L^2(\mathbb{R})$ be such that $\mathcal{G}(g, \Lambda)$ is a Parseval frame for $L^2(\mathbb{R})$, e.g., $g = \chi_{[0,1)}$. Then also $\mathcal{G}(g, (2\mathbb{Z}+1) \times \frac{1}{2}\mathbb{Z})$ is a Parseval frame for $L^2(\mathbb{R})$, since

$$\sum_{m,n\in\mathbb{Z}} \left|\left\langle f, \rho(2m+1, \tfrac{1}{2}n)g \right\rangle\right|^2 = \sum_{m,n\in\mathbb{Z}} \left|\left\langle \rho(-1,0)f, \rho(2m, \tfrac{1}{2}n)g \right\rangle\right|^2 = \|\rho(-1,0)f\|_2^2 = \|f\|_2^2$$

for all $f \in L^2(\mathbb{R})$. We observe that we can decompose Λ_{SI} as $((2\mathbb{Z}+1)\times\frac{1}{2}\mathbb{Z}) \cup (2\mathbb{Z}\times\frac{1}{2}\mathbb{Z})$. Since each function in the system $\mathcal{G}(g, \Lambda_{SI}, w)$ is equipped with the weight $\frac{1}{\sqrt{2}}$, also $\mathcal{G}(g, \Lambda_{SI}, w)$ forms a Parseval frame for $L^2(\mathbb{R})$. Thus $(\Lambda, \Lambda_{SI}, w, \chi_{[0,1)})$ is a shift-invariant extension pair with respect to the frame bound 1.

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References

[1] G. Ascensi and G. Kutyniok, *Accumulative density*, in "Wavelets XI" (San Diego, CA, 2005), SPIE Proc., 5914, SPIE, Bellingham, WA (2005), 188–195.

[2] R. Balan, P. G. Casazza, C. Heil, and Z. Landau, *Density, overcompleteness, and localization of frames. II. Gabor systems*, preprint (2005).

[3] O. Christensen, B. Deng, and C. Heil, *Density of Gabor frames*, Appl. Comput. Harmon. Anal. **7** (1999), 292–304.

[4] C. K. Chui, X. Shi, and J. Stöckler, *Affine frames, quasi-affine frames, and their duals*, Adv. Comp. Math. **8** (1998), 1–17.

[5] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, (1992).

[6] H. G. Feichtinger, K. Gröchenig, and T. Strohmer, *Efficient numerical methods in non-uniform sampling theory*, Numer. Math. **69** (1995), 423–440.

[7] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[8] C. Heil, *An introduction to weighted Wiener amalgams*, in "Wavelets and their Applications" (Chennai, January 2002), M. Krishna, R. Radha, and S. Thangavelu, eds., Allied Publishers, New Delhi (2003), 183–216.

[9] C. Heil and G. Kutyniok, *Density of frames and Schauder bases of windowed exponentials*, preprint (2005).

[10] E. Hernández, D. Labate, and G. Weiss, *A unified characterization of reproducing systems generated by a finite family, II*, J. Geom. Anal. **12** (2002), 615–662.

[11] G. Kutyniok, *Affine density, frame bounds, and the admissibility condition for wavelet frames*, Constr. Approx., to appear.

[12] H. Landau, *Necessary density conditions for sampling and interpolation of certain entire functions*, Acta Math. **117** (1967), 37–52.

[13] J. Ramanathan and T. Steger, *Incompleteness of sparse coherent states*, Appl. Comput. Harmon. Anal. **2** (1995), 148–153.

[14] A. Ron and Z. Shen, *Weyl-Heisenberg frames and Riesz bases in $L_2(\mathbb{R}^d)$*, Duke Math. J. **89** (1997), 237–282.

Institute of Mathematics, Justus–Liebig–University Giessen, 35392 Giessen, Germany
E-mail address: `gitta.kutyniok@math.uni-giessen.de`