PROMP: A Sparse Recovery Approach to Lattice-Valued Signals

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Abstract

Applications such as wireless communications require efficient sensing techniques of signals with the a priori knowledge of those being lattice-valued. In this paper, we study the impact of this prior information on compressed sensing methodologies, and introduce and analyze PROMP ("PReprojected Orthogonal Matching Pursuit") as a novel algorithmic approach for sparse recovery of lattice-valued signals. More precisely, we first show that the straightforward approach to project the solution of Basis Pursuit onto a prespecified lattice does not improve the performance of Basis Pursuit in this situation. We then introduce PROMP as a novel sparse recovery algorithm for lattice-valued signals which has very low computational complexity, alongside a detailed mathematical analysis of its performance and stability under noise. Finally, we present numerical experiments which show that PROMP outperforms standard sparse recovery approaches in the lattice-valued signal regime.

Keywords: Compressed Sensing, Basis Pursuit, Lattice Search, Orthogonal Matching Pursuit, High-dimensional Geometry.

1 Introduction

During the last 10 years, the area of compressed sensing or, more generally, sparse recovery has matured to a novel research area intersecting, in particular, mathematics, computer science, and electrical engineering. Its main objective is to efficiently solve underdetermined linear equations

$$Ax = b,$$

with $A \in \mathbb{R}^{m,d}$ and $b \in \mathbb{R}^m$, $m < d$ under the additional assumption that the solution $x \in \mathbb{R}^d$ is *sparse*. The feasibility of this hypothesis is by now generally accepted, and sparsity of data can be identified as a new paradigm in signal and image processing. The most basic notion of sparsity of a vector $x = (x_1, \dots, x_d)$ states that x is called k-*sparse* provided that the number of non-zero elements x_i is less than or equal to k. In this situation, sufficient conditions – typically in terms of incoherence properties of the measurement matrix A and the sparsity of x – for precise recovery of the signal b by, for instance, convex optimization algorithms are known even when the measurements are contaminated with noise. We refer the interested reader to [11] for a survey.

In many applications however, additional information of the original signal is known and should be exploited to improve recovery guarantees. In this paper, we will focus on the situation of lattice-valued signals, which appear, for instance, in massive MIMO [35], wideband spectrum sensing [2] and error correcting codes [5]. Previous work on sparse recovery aspects of this situation has however mostly focused on integer-valued or binary signals and without a detailed mathematical analysis of recovery guarantees. Hence, a sparse recovery algorithm for lattice-valued signals alongside a precise analysis of its performance also under noise is still an open problem.

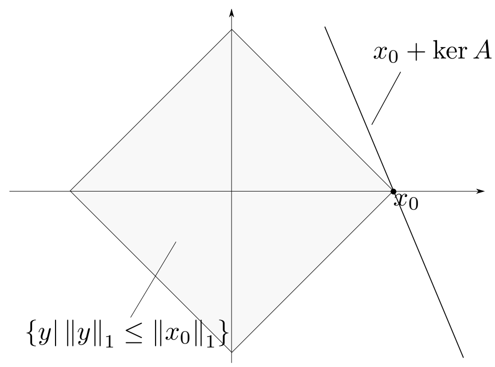


Figure 1: Sparse signals x_0 lie on low-dimensional faces of the ℓ_1 unit ball.

1.1 Algorithms for Sparse Recovery

The most standard approach to the sparse recovery problem is the Basis Pursuit algorithm, [8] or rather method, which consists of searching for the solution to the equation $Ax = b$ having the smallest ℓ_1-norm, i.e.,

$$\|x\|_1 \text{ s.t. } Ax = b. \tag{$\mathcal{P}_1$}$$

A beautiful geometrical intuition stands behind this approach: Sparse vectors x_0 with unit ℓ_1-norm lie on low-dimensional faces of the cross polytope $T^d = \{x \mid \|x\|_1 = 1\}$. Thus it is probable that the set $x_0 + \ker A$ only touches T^d in x_0 (Figure 1). By using random matrices such as Gaussian iid, it is possible to achieve a very high recovery probability given that the number of measurements satisfies $m \gtrsim s \log(d)$, where s is the sparsity of the signal x_0 [6].

The sparse recovery algorithm Orthogonal Matching Pursuit (Algorithm 1) is of different nature [34] . Its objective is to iteratively construct a support estimate S. In each step, one greedily chooses a new index i to minimize

$$\min_{\operatorname{supp} v \subseteq S \cup i} \|Av - b\|_2 .$$

This algorithm also requires an order of $s \log(d)$ measurements to succeed at recovering an s-sparse vector x_0 [43]. Empirically, it performs slightly worse than Basis Pursuit when it comes to recovery probabilities. Main advantages of Orthogonal Matching Pursuit are though that this algorithm is very fast and easy to implement. [42]. Since this algorithm will be a backbone of the algorithmic approach being developed in our paper, we briefly state its pseudo-code version.

Algorithm 1: *OMP* – Ortogonal Matching Pursuit

Data: A matrix $A \in \mathbb{R}^{m,d}$ and a vector $b \in \mathbb{R}^m$.
Result: An estimate x of a sparse solution to $Ax = b$.
Initalize $x = 0$, $S = \emptyset$ and $\rho = b$.
while $Ax \neq b$ **do**
  Calculate $j = \operatorname{argmax}_i |\langle a_i, \rho \rangle|$.
  Update $\hat{S} = \hat{S} \cup \{j\}$.
  Update $x = \operatorname{argmin}_{\operatorname{supp} v \subseteq \hat{S}} \|Av - b\|$ and $\rho = b - Ax$.

Certainly, a variety of other algorithm performing sparse recovery are available, and we refer to [11] for a survey. Since in the sequel Basis Pursuit and Orthogonal Matching Pursuit will play the key roles, we for now refrain from detailing other approaches.

1.2 Models of Sparse Signals

To accommodate and also utilize specific priors from applications for the sparse signals to be recovered, various models of sparse signals have been introduced. Let us briefly recall some of those closest to our approach, without any claim of completeness.

We start with highlighting structural assumptions on the set of sparse signal, which were incorporated into the classical Basis Pursuit algorithm by suitable modifications. One important case are binary signals leading the framework of 1-bit Compressed Sensing [4]. Another popular assumption is *block sparsity* of the signal. In this case, a modified version of Basis Pursuit leads to better reconstruction properties than for general signals [39]. A slightly different prior is the assumption that certain parts of the possible index set $\{1, \dots d\}$ are more probable to be part of the support of the solution vector than others. A proposed method to account for this, which has been thoroughly investigated is *weighted ℓ_1-minimization* [19, 24, 33].

While modifications of Basis Pursuit are much more popular, there have also been approaches to incorporate prior knowledge about a specific model of sparse signals into greedy approaches. In this regime we would like to highlight the situation of a known support estimate for the original sparse signals discussed before, in which a variant of OMP was warm-started [7]. Variants of this approach – yet still following the same idea – are contained, for instance, in [40].

1.3 Lattice-valued Signals

In this paper, we will focus on the structural assumption that the sparse signal is lattice-valued. More precisely, we assume that the original signal $x_0 \in G\mathbb{Z}^d$ with $G \in \mathbb{R}^{d,d}$ an invertible matrix. The sparsity constraint will be implemented by considering the situation when x_0 possesses a sparse representation in the lattice $G\mathbb{Z}^d$, i.e., when

$$x_0 = Gv_0 \quad \text{where } v_0 \in \mathbb{Z}^d \text{ is a } \textit{sparse} \text{ vector}$$

in the sense of v_0 having few non-zero entries. If v_0 is s-sparse, i.e., the number of non-zero entries in v_0 is at most s, we will refer to x_0 as being *s-G-sparse*.

This class of signals appears naturally in wireless communications due to the fact that typically signals are quantized before being transmitted. Often, the signals are even bit sequences, i.e., members of the set $\{0, \pm 1\}^d$. Furthermore, there is several applications of wireless communications compressed sensing or, more generally, sparse recovery plays an increasingly important role, including massive MIMO [35], wideband spectrum sensing [2], and error correcting codes [5]. Therefore, there is a pressing need to specialize compressed sensing techniques to lattice-valued signals.

1.4 Previous Work

The problem of developing a sparse recovery algorithm for lattice-valued sparse signals has already been considered by several authors, however mostly focusing on integer-valued or binary signals and without a detailed mathematical analysis of recovery guarantees. Let us briefly report on the different approaches which have been suggested.

One line of research has been to model compressed sensing as a graph-theoretical problem [31, 45]. Put shortly, one constructs a bipartite graph consisting of nodes $C_i, i = 1, \dots d$ representing the entries of the sought for vector x_0 and nodes V_i representing the entries of the vector $b = Ax$. An edge is then drawn between V_i and C_j if $a_{ij} \neq 0$. The basic idea consists of identifying V-nodes which are only connected to one C-node, and iteratively removing the corresponding edges and nodes. Although the approach certainly is interesting, it will only work provided the graph contains only relatively few edges, i.e., in case the matrix A is sparse. In many applications, this assumption is however not very realistic such as, for instance, in wireless communications, where the matrix is often assumed to be Gaussian. Moreover, the beautiful geometric intuition behind compressed sensing is totally lost in this approach.

A different approach, which is not constrained to the setting of binary signals but instead considers signals with entries from a finite alphabet $\mathbb{F} = \{0, 1, \dots p\}$ is considered, for instance, the papers [15, 41]. A major difference from compressed sensing is that $\mathbb{F}$ is not only regarded as a set, but in fact as a field (i.e., p is assumed to be prime). Hence, the sensing matrix A is chosen from $\mathbb{F}^{m,d}$ and all operations are made modulo p. This is indeed very intriguing, but after all relatively far from the problem we are considering in this article.

An important algorithm for general recovery of integer signals from noisy linear measurements is the *sphere decoder* [1]. There have been several attempts to adapt the sphere decoder to sparse signals such as in [41, 47]. This leads to algorithms with more robustness to noise and which are faster. However, the case of underdetermined systems is still relatively problematic. There do exist a few papers dealing with this situation as well: The authors [44] suggest to artificially add extra equations to the system. They show that this approach will work for so called constant-modulus signaling schemes. It is probably not possible to achieve a good performance also for the sparse setting which we are considering. Another philosophy was presented in [9]. The idea of that paper is to combine sphere decoding of a part of length m of the signal with a brute force search over the rest of the signal. This causes the complexity of the computations to grow immensely, if $d \gg m$. As of today, more sophisticated ways of determining the remaining part of the signal have been developed, see for instance [46], but the methods remain relatively heuristic, and theoretical performance guarantees are rare.

Yet another approach which has been considered is to use variants of convex optimization algorithms. The paper [29] deals with the problem of recovering a signal in $\{\pm 1\}^d$ from linear measurements using ℓ_∞-minimization:

$$\min \|x\|_\infty \text{ subject to } Ax = b.$$

The authors are able to derive the intriguing result that already as few as $\frac{d}{2}$ measurements will be sufficient to almost certainly recover any signal in $\{\pm 1\}^d$. This high success probability is although heavily dependent on the fact that the signals considered do not contain any zeros. This has the consequence that all entries in the vector have the same absolute value. Such vectors are called *saturated* or *democratic*, and for such vectors, ℓ_∞-minimization is perfectly suited [20]. Hence, although the results of this paper are a very interesting contribution, its results are not applicable to sparse signals.

A clever modification of the basis pursuit procedure to solve the problem of sparse recovery of *positive* binary signals, namely to add the constraint that each entry of the vector should lie between 0 and 1, e.g.

$$\min \|x\|_1 \text{ subject to } Ax = b, \forall\, i \in \{1, \ldots, d\} : 0 \leq x(i) \leq 1.$$

was suggested in [14, 38]. This modification indeed leads to significantly better recovery probabilities. For instance, for sparsity levels s greater than $\frac{d}{2}$, only s measurements will suffice for the recovery of an s-sparse signal. The positivity constraint is thereby not of major importance, since the constraint can be relaxed to $\forall\, i \in \{1, \ldots, d\} : -1 \leq x(i) \leq 1$ to incorporate general binary signals. The only problem of this approach that it does not work for more general integer-valued signals.

1.5 The PROMP Algorithm

In this paper, which builds on the master thesis of one of the authors [17], we investigate the problem of sparse recovery of lattice-valued signals on the one hand using convex optimization and on the other hand a greedy approach. The main representatives of those two classes of algorithms, namely Basis Pursuit and Orthogonal Matching Pursuit, will serve as our starting points.

First, we proclaim and analyze a very natural modification of Basis Pursuit to yield better recovery results for sparse lattice-valued signals, namely what we coin *Basis Pursuit with Post-Projection.* In this approach, first Basis Pursuit is performed, i.e., an initial signal estimate $\hat{x}$ is computed using $\mathcal{P}_1$, followed by a projection onto the respective lattice $G\mathbb{Z}^d$, where G is an invertible $\mathbb{R}^{d,d}$-matrix. With Theorems 2.1 and 2.2 we surprisingly show that for most lattices, including the important case of $\mathbb{Z}^d$, there does not exist a measurement matrix $A \in \mathbb{R}^{m,d}$ satisfying the following condition: There exists a signal x_0 which is exactly recovered from Ax_0 by Basis Pursuit with Post-Projection, but the initial signal estimate $\hat{x}$ from Basis Pursuit differs from x_0. Stating this concisely, one can say that *post-projection is redundant.*

Second, we pursue a different strategy, namely to adapt Orthogonal Matching Pursuit to incorporate the prior information of the signal to be lattice-valued. This general tactic was already pursued in the papers [36, 37]. The authors of those papers altered the calculation of the index which is added to the support approximation (line 2 of Algorithm 1), using both lattice projection algorithms, e.g., sphere decoding and soft-feed back techniques arising from a Bayesian approach.

We will do something radically different than the technique from [36, 37]. Instead of altering the Orthogonal Matching Pursuit Algorithm itself, we will use a *Support Approximation Step* to obtain an approximate idea of the support of the signal x_0 somehow in the spirit of preconditioning. This Support Approximation Step consists of performing the ℓ_2-minimization

$$\min \|x\|_2 \text{ subject to } Ax = b, \tag{$\mathcal{P}_2$}$$

and then forming a support estimate by hard thresholding:

$$S_\vartheta = \left\{ i \mid |\hat{x}(i)| \geq \frac{m}{d}\vartheta \right\},$$

where $\hat{x}$ is the solution of $\mathcal{P}_2$ and $\vartheta \in (0,1)$ is some parameter. It might come as a surprise that we use ℓ_2-minimization. However, the use of ℓ_2-minimization has several advantages. Firstly, it is easy to implement and can be computed quickly. And, secondly, the solution can be written down explicitly as

$$\hat{x} = \Pi_{\ker A^\perp} x_0,$$

which allows for a rich theoretical study of its performance, also because of the fact that it is geometrically very intuitive. In the second step of the algorithm, we then use S_θ to warm-start OMP, hence coining it *Warm-Starting OMP Step.*

Algorithm 2 shows both steps in more detail. To highlight the idea of pre-projection as a way to approximate the support set, we name is $PROMP$ (PReprojected Orthogonal Matching Pursuit). We wish to emphasize that in this paper, we concentrate on the Gaussian case for the measuring process. We however expect the method to work for other measurement matrices as well, in particular, for partial samples of orthonormal bases.

Algorithm 2: $PROMP$ – PRe-projected Ortogonal Matching Pursuit

Data: A matrix $A \in \mathbb{R}^{m,d}$ and a vector $b \in \mathbb{R}^m$.
Result: An estimate x of a sparse, discrete solution of $Ax = b$.
Calculate $w = \operatorname{argmin} \|u\|_2$ s. t. $Au = b$.
if $Aw = b$ **then**
  return $x = w$; /* ℓ_2-min. alone was successful. */
else
  Calculate the indices i for which $|w_i| \cdot d/m \geq 1/2$. Form the set S_ϑ of such indices.
Set $x = \operatorname{argmin}_{\operatorname{supp} v \subseteq S_\vartheta} \|Av - b\|$ and $\rho = b - Ax$, and initalize $S = S_\vartheta$.; /* Run OMP initialized with S_ϑ. */
while $Ax \neq b$ **do**
  Calculate $j = \operatorname{argmax}_i |\langle a_i, \rho \rangle|$.
  Update $S = S \cup j$.
  Update $x = \operatorname{argmin}_{\operatorname{supp} v \subseteq \hat{S}} \|Av - b\|$ and $\rho = b - Ax$.

In this paper, we will provide a novel type of analysis on the impact of the warm starting procedure we propose to the performance of Orthogonal Matching Pursuit. This analysis is independent of the concrete procedure of building the support approximation, and is hence interesting in its own right. We are able to prove that in the case of a Gaussian measurement matrix, the number of measurement vectors needed will be significantly reduced provided that the support estimate is fairly correct. In particular, the performance may be improved although some indices not contained in the support of x_0 are included in the initial support approximation.

Concerning our choice of the Support Approximation Step, we should mention that in general ℓ_2-minimization as a mean of sparse recovery has been used very sparsely in the literature. After all, the solution of the problem $\mathcal{P}_2$ will typically be non-sparse, as is well known. Indeed, one can easily convince oneself that the probability that the solution of P_2 is equal to x_0 is zero for the case that A is Gaussian. One exception that should be mentioned in this context, although it is not directly related to our work, is the method of *iteratively reweighted least squares* [10]. Here, one alters the ℓ_2-norm using weights, which are iteratively adapted to the signal in order for it to mimic the behaviour of the ℓ_1, or even the non-convex ℓ_p-norm for $p \in (0, 1)$. We will though see that in our case, ℓ_2-minimization serves perfectly as an approximate procedure for deriving a meaningful support estimate.

1.6 Our Contribution

Our contribution is hence four-fold. First, we provide a detailed analysis that the most natural approach for sparse recovery of lattice-valued signals, which is to apply Basis Pursuit followed by projection onto the prespecified lattice of the original signal, is completely redundant. Second, we introduce with PROMP a novel algorithm designed for lattice-valued signals. Numerical experiments show that PROMP outperforms standard approaches such as Basis Pursuit in the lattice-valued signal regime. Advantages of PROMP are also its low computational complexity and the fast that it is very easy to implement. Third, the preprocessing step of PROMP can also be exploited for other sparse recovery algorithms leading to improved performance for lattice-valued signals. We anticipate that our idea of preprocessing might lead to the development of other preprocessing steps adapted to the structural preknowledge of the original signal. And, fourth, we provide a detailed mathematical analysis of the sparse recovery guarantees of PROMP also under noise. This analysis, as mentioned before, is independent of the concrete procedure of building the support approximation, and might be also interesting for the analysis of similar algorithmic approaches.

1.7 Outline

The paper is organized as follows. The post-projection approach is presented and analyzed in Section 2. A detailed analysis of the PROMP algorithm is provided in Section 3 with Subsection 3.1 containing results on the accuracy of the selected support set and a stability analysis of the Support Approximation Step and Subsection 3.2 proving successful recovery by the Warm-Starting OMP Step. Section 4 is then devoted to various numerical experiments both in the exact and noisy regime as well as concerning comparison to other approaches. Since the proofs of several results are rather technical in nature, the last Section 5 contains those outsourced proofs.

2 Post-Projection is Redundant

Let us begin by fixing the model situation. We will always assume that we are given m linear measurements of the d-dimensional signal x_0, expressed through a matrix $A \in \mathbb{R}^{m,d}$. The signal x_0 is assumed to lie on a lattice $\Lambda = G\mathbb{Z}^d$, where $G \in \mathbb{R}^{d,d}$ is an invertible matrix. We will often in particular consider the situation when x_0 has a sparse representation in the lattice, i.e., when there exists a *sparse* vector $v_0 \in \mathbb{Z}^d$ so that $x_0 = Gv_0$. If v_0 is s-sparse, we will refer to x_0 as being *s-G-sparse*.

Now assume that x_0 is s-G-sparse and we acquire measurements of the form Ax_0. In this situation, a well-known reasonable approach to reconstruct x_0 would be to use Basis Pursuit [8], i.e., to solve

$$\min \|v\|_1 \quad \text{subject to } AGv = b, \qquad (\mathcal{P}_{1,G})$$

where $b = Ax_0$, aiming to find an estimate for v_0, say $\hat{v}$. Then use $\hat{x} = G\hat{v}$ as an estimate for x_0. This approach does, however, not at all use the fact that $v_0 \in \mathbb{Z}^d$ in the sense of using the information that the original signal v_0 is integer valued. The perhaps most reasonable and simple way to incorporate this prior information is to choose the point on the lattice Λ, which is the closest to $\hat{x}$, as the estimate of x_0, i.e., to *post-project*. Thus we now consider *Basis Pursuit with Post-Projection* given by

$$\Pi_{G\mathbb{Z}^d}(\operatorname{argmin} \|v\|_1 \quad \text{subject to } AGv = b),$$

where $\Pi_{G\mathbb{Z}^d}$ denotes the orthogonal projection to the closest element in the lattice $G\mathbb{Z}^d$ in Euclidian distance. Notice that this element might not be unique, and application of $\Pi_{G\mathbb{Z}^d}$ could yield a set. We now ask: Does this enhance the recovery probability?

In order to investigate this question, let us begin by noting that the output of the procedure described above will be equal to the ground truth signal x_0 if and only if $\hat{x}$ lies in the *Voronoi region* $\Omega(G\mathbb{Z}^d, x_0)$ of x_0 in $G\mathbb{Z}^d$ [1], which is

$$\Omega(G\mathbb{Z}^d, x_0) = \left\{ v \mid \forall x \in G\mathbb{Z}^d : \|v - x_0\|_2 \leq \|v - x\|_2 \right\}.$$

This is in turn the case precisely when $\hat{v} \in G^{-1}\Omega(G\mathbb{Z}^d, x_0) = v_0 + G^{-1}\Omega(G\mathbb{Z}^d, 0)$. For convenience, let us define

$$P_G := G^{-1}\Omega(G\mathbb{Z}^d, 0).$$

We will see that the structure of P_G actually determines if it is at all possible that a signal v_0, which is *not* reconstructed by Basis Pursuit directly, will be reconstructed by Basis Pursuit with Post-Projection. Let us start by proving a relatively simple result in this direction, which shows that for certain lattices Basis Pursuit with Post-Projection does in fact *not* improve recoverability.

Theorem 2.1. *Let $v_0 \in \mathbb{Z}^d$ and $x_0 = Gv_0$, where $G \in \mathbb{R}^{d,d}$ is an invertible matrix. Suppose that the lattice $G\mathbb{Z}^d$ has the property that*

$$P_G \subseteq (-1, 1)^d. \tag{1}$$

If all solutions of $(\mathcal{P}_{1,G})$ lie in $v_0 + P_G$, then all those solutions are equal to v_0.

Proof. Since the case $x_0 = 0$ is trivial, we may without loss of generality assume that $x_0 \neq 0$. Next assume, towards a contradiction, that all solutions of $(\mathcal{P}_{1,G})$ with $b = Ax_0$ are contained in $v_0 + P_G$, and that there exists at least one solution, say v, which is not equal to v_0. First, define the vector

$$\hat{1}(i) = \begin{cases} 1 & \text{, if } v(i) > 0 \\ -1 & \text{, if } v(i) < 0 \\ 0 & \text{, else.} \end{cases}$$

Then we have $\|v\|_1 = \langle \hat{1}, v \rangle$. Since $P_G \subseteq (-1,1)^d$ and all non-zero entries of v_0 have absolute value at least 1, v and v_0 have the same sign pattern on $\operatorname{supp} v_0$. Therefore, we also have $\|v_0\|_1 = \langle \hat{1}, v_0 \rangle$, and consequently $\langle \hat{1}, v - v_0 \rangle = \|v\|_1 - \|v_0\|_1 \leq 0$. Now, let $\epsilon > 0$, and consider the vector

$$v_\epsilon := v + \epsilon(v - v_0).$$

The geometry of the following arguments is illustrated in Figure 2. It is immediately clear that $Av_\epsilon = Av = Av_0$ and $\langle \hat{1}, v_\epsilon \rangle \leq \langle \hat{1}, v \rangle$. This means that if we can choose ϵ so that $v_\epsilon \notin v_0 + P_G$ but still $\|v_\epsilon\|_1 = \langle \hat{1}, v_\epsilon \rangle$, the resulting vector v_ϵ would be a solution of $(\mathcal{P}_{1,G})$ which does not lie in $v_0 + P_G$; a contradiction. We now argue that such an ϵ always exists.

First observe that $\langle \hat{1}, v_\epsilon \rangle = \|v_\epsilon\|_1$ if and only if v_ϵ has the same sign pattern as v. This is true for any $\epsilon > 0$, provided that there does not exist some i for which the signs of $v(i)$ and $v(i) - v_0(i)$ differ. In that case we can thus choose ϵ in such a way that $\|v_\epsilon - v_0\|_\infty \geq 1$. By assumption on P_G, this implies $v_\epsilon \notin v_0 + P_G$.

If however there does exist some i such that $v(i) - v_0(i)$ and $v_0(i)$ do have different signs, then the signs of $v_\epsilon(i)$ and $v_0(i)$ will be different for large values of ϵ. Hence, the entity

$$\epsilon^* = \sup_{\epsilon > 0} \left\{ \langle \hat{1}, v_\epsilon \rangle = \|v_\epsilon\|_1 \right\}$$

will be smaller than ∞. Due to continuity, it follows that $\langle \hat{1}, v_{\epsilon^*} \rangle = \|v_{\epsilon^*}\|_1$. Since $\langle \hat{1}, v_\epsilon \rangle \neq \|v_\epsilon\|_1$ for values of ϵ slightly larger than ϵ^*, there must exist some i_0 so that $v_\epsilon(i_0)$ changes sign as ϵ surpasses ϵ^*. In particular, $v_{\epsilon^*}(i_0) = 0$.

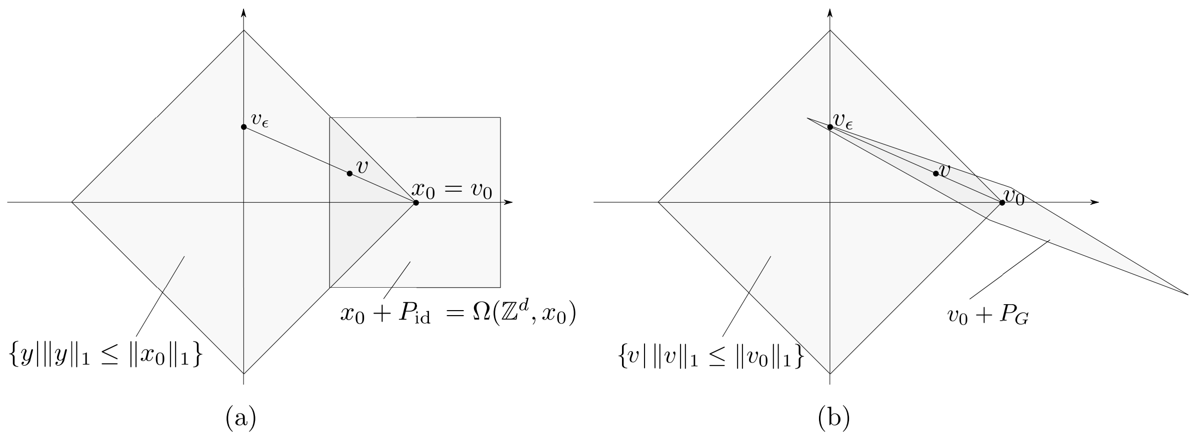


Figure 2: Illustration for the proof of Theorem 2.1. (a): Situation of the standard lattice $\mathbb{Z}^d$, which indeed fulfills $P_G \subseteq (-1,1)^d$. (b): Situation for a lattice $G\mathbb{Z}^d$ which does not fulfill $P_G \subseteq (-1,1)^d$.

If $v_0(i_0) = 0$, then necessarily $v(i_0) \neq 0$, since otherwise $v_\epsilon(i_0)$ will be constantly equal to zero, contradicting the change of sign in ϵ^*. But if $v(i_0) \neq 0$, then $v_{\epsilon^*}(i_0) = (1+\epsilon^*)v(i_0) \neq 0$, which is a contradiction. Therefore, $v_0(i_0) \neq 0$, and consequently

$$\|v_{\epsilon^*} - v_0\|_\infty \geq |v_{\epsilon^*}(i_0) - v_0(i_0)| = |v_0(i_0)| \geq 1,$$

since $v_0 \in \mathbb{Z}^d$. Again, by assumption on P_G, as before this implies $v_\epsilon \notin v_0 + P_G$. The proof is finished. □

The above theorem already covers the important case of the standard lattice, i.e., that $G = I$. In this case, we have $P_G = \Omega(\mathbb{Z}^d, 0) = [-1/2, 1/2]^d$, which is certainly contained in $(-1,1)^d$. Hence, in the case of the standard lattice, Post-Projection will *never* enhance the recovery probability of Basis Pursuit!

It is, however, possible to further restrict the lattices for which Post-Projection has a chance of 'helping' Basis Pursuit, which is the content of our next result. Part (i) studies sufficient conditions on the lattice $G\mathbb{Z}^d$ such that Post-Projection never improves a unique solution. Part (ii) provides sufficient conditions on $G\mathbb{Z}^d$ for a measurement matrix and a sparse vector to exist such that Post-Projection does help.

Theorem 2.2. *Let $v_0 \in \mathbb{Z}^d$ and let $G \in \mathbb{R}^{d,d}$ be an invertible matrix.*

(i) Suppose that, for each $1 \leq \ell \leq s$, P_G does not contain any vectors of the form $v + n$, for which

(a) the supports of v and n are disjoint, and

(b) v is m-sparse and $n \in \mathbb{Z}^d$ has exactly ℓ non-zero entries.

Then, for every s-G-sparse $x_0 \in G\mathbb{Z}^d$, there do not exist any matrices $A \in \mathbb{R}^{m,d}$ so that $(\mathcal{P}_{1,G})$ with $b = Ax_0$ has a unique solution $\hat{v} \neq v_0$ satisfying $\hat{v} \in v_0 + P_G$.

(ii) Suppose that there exists some $\ell \geq 2$ and a vector of the form $v + n \in P_G$, for which v and n fulfill (a) and (b). Then, for every m with $m + \ell \leq d$, there exists a matrix $A \in \mathbb{R}^{m,d}$ and an ℓ-G-sparse vector $x_0 \in G\mathbb{Z}^d$ such that $(\mathcal{P}_{1,G})$ with $b = Ax_0$ has a vector $\hat{v} \neq v_0$ as solution satisfying $\hat{v} \in v_0 + P_G$.

The proof of this result, which uses a geometrical characterization of matrices allowing signal recovery with Basis Pursuit due to Donoho and Tanner [13], is postponed to the last section.

3 Analysis of the $PROMP$ algorithm

The important lesson of the last section is that the strategy of Post-Projection after Basis Pursuit is too simple to really make use of the assumption that x_0 lies on a lattice additional to being sparse. Therefore, we will in the rest of the paper concentrate of the $PROMP$ Algorithm 2, which was presented already in the introduction. Let us briefly recall its two steps.

1. *Support Approximation Step.*
   For given $A \in \mathbb{R}^{m,d}$ and $b \in \mathbb{R}^d$, calculate

$$\hat{x} := \operatorname{argmin}_x \|x\|_2 \quad \text{subject to } Ax = b, \tag{$\mathcal{P}_2$}$$

   and use this solution to construct an approximation of the support of original signal x_0 through

$$S_\vartheta = \left\{ i \mid |\hat{x}(i)| \geq \frac{m}{d}\vartheta \right\} \tag{2}$$

   with $\vartheta \in (0,1)$ carefully chosen.

2. *Warm-Starting OMP Step.*
   Run OMP initialized with the support approximation S_ϑ.

We will subsequently present a performance analysis of each of these steps. They will be independent, and hence, they are also separately interesting. We will always assume that the measurement matrix A is random, with the standard Gaussian distribution. This will be crucial in many arguments, and it should at this place be explicitly noted that it probably is not trivial to generalize the results to other distributions.

3.1 Analysis of Support Approximation Step

We will divide the analysis of this step into two parts. First, we will focus on the accuracy of the selected support set dependent on the number of measurements. Second, we will drive a stability analysis of this step.

3.1.1 Classification of Indices using ℓ_2-Minimization

Aiming to show that provided m is large enough, the Support Approximation Step of $PROMP$ will produce a reasonable estimation of $\operatorname{supp} x_0$, we will prove that, with high probability, $S_\vartheta \cap \operatorname{supp} x_0$ will be large at the same time as $S_\vartheta \cap (\operatorname{supp} x_0)^c$ is small. The idea will be to investigate the distribution of the random variable $\hat{x}$. Then we will use a concentration of measure argument to argue that for each $i \in [1, \ldots, d]$, we have $\hat{x}(i) \approx m/d \cdot x_0(i)$ with high probability. This has the consequence that only the entries corresponding to indices $i \in \operatorname{supp} x_0$ can be large in magnitude, which implies that S_ϑ is a reasonable approximation of $\operatorname{supp} x_0$.

We have already noted that $\hat{x} = \Pi_{\ker A^\perp} x_0$, where due to the Gaussianity of A, $\ker A^\perp$ is uniformly distributed over the Grassmannian manifold of m-dimensional subspaces of $\mathbb{R}^d$, in symbols $\ker A^\perp \sim \mathcal{U}(G(d,m))$. It is thus of major interest to investigate the distribution of $\Pi_L x_0$, where $L \sim \mathcal{U}(G(d,m))$. The following lemma provides a decomposition of that distribution, which is also interesting in its own right.

Lemma 3.1. *Let $d \geq 2$ and $k \leq d$, fix $x_0 \in \mathbb{S}^{d-1}$ and let $L \sim \mathcal{U}(G(d,k))$. Then*

$$\Pi_L x_0 = R^2 x_0 + R\sqrt{1-R^2}\, Q_{x_0}[\theta, 0],$$

where

- *R is distributed as the norm of $\Pi_L x_0$.*
- *$\theta \sim \mathcal{U}(\mathbb{S}^{d-2})$, independent of R. $[\theta, 0] \in \mathbb{R}^d$ denotes the vector formed when extending θ with a zero element. Also note that we understand $\mathbb{S}^0$ as the set $\{\pm 1\}$.*

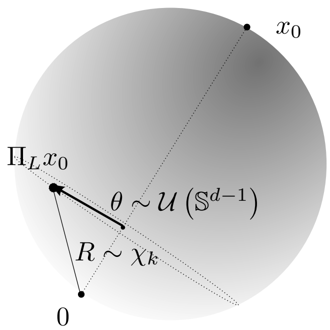


Figure 3: Illustration of the decomposition in Lemma 3.1.

- *Q_{x_0} is any fixed orthogonal matrix with $Q_{x_0} e_d = x_0$.*

Proof. Let us first note that $\Pi_L x_0$ is always an element of the sphere centered in $x_0/2$ with radius $1/2$. Indeed, if $(q_1, \dots, q_k)$ is an orthonormal basis of L, and $(q_{k+1}, \dots q_d)$ is one of $L^\perp$, we have

$$\|\Pi_L x_0 - x_0/2\|_2^2 = \left\| \frac{1}{2} \sum_{i=1}^{k} \langle x_0, q_i \rangle \, q_i - \frac{1}{2} \sum_{i=k+1}^{d} \langle x_0, q_i \rangle \, q_i \right\|_2^2 = \frac{1}{4} \|x_0\|_2^2 = \frac{1}{4}.$$

Due to the fact that $Q_{x_0} e_d = x_0$, we have $\langle x_0, Q_{x_0}[\Theta, 0] \rangle = \langle e_d, [\Theta, 0] \rangle = 0$ for every $\Theta \in \mathbb{S}^{d-2}$. Hence, if we write $\Pi_L x_0 = \lambda x_0 + \mu Q_{x_0}[\Theta, 0]$ for some λ, μ and Θ, we have

$$\|\Pi_L x_0\|_2^2 = \lambda^2 + \mu^2,$$
$$1/4 = \|\Pi_L x_0 - x_0/2\|_2^2 = (\lambda - 1/2)^2 + \mu^2.$$

Denoting $R := \|\Pi_L x_0\|_2$, this system of equations only has the solutions

$$\lambda = R^2 \quad \text{and} \quad \mu = \pm R\sqrt{1 - R^2}.$$

Therefore, if we define the map $\Lambda : [0,1] \times \mathbb{S}^{d-2} \to \mathbb{R}^d, (\rho, \Theta) \to \rho^2 x_0 + \rho\sqrt{1-\rho^2} Q_{x_0}[\Theta, 0]$, we will have $\Pi_L x_0 = \Lambda(R, \theta)$ for some variable θ distributed on the sphere.

It only remains to prove that R and θ are independent, and that $\theta \sim \mathcal{U}\left(\mathbb{S}^{d-2}\right)$. For this, it suffices to prove that for each pair of Borel sets $A \subseteq [0,1]$ and $B \subseteq \mathbb{S}^{d-2}$ with probability $\mathbb{P}(R \in A) > 0$, we have

$$\frac{\mathbb{P}(R \in A, \theta \in B)}{\mathbb{P}(R \in A)} = \sigma^{d-2}(B), \tag{3}$$

where $\mathbb{P}$ denotes the distribution of $\Pi_L x_0$ and σ^{d-2} the standard normalized surface measure of the sphere $\mathbb{S}^{d-2}$. It is clear that for each fixed A, the left hand side of (3) defines a probability measure on $\mathbb{S}^{d-2}$. It is enough to prove that this measure is uniformly distributed, since this uniqely defines the measure σ^{d-2} (for a definition of what is meant by uniformity of a distribution and a proof of the claim, see for instance [25, p.88]).

Towards this end, we first prove a symmetry of Π. If $q \in O(d)$ is such that $qx_0 = x_0$, we have for every $M \subseteq \mathbb{R}^d$

$$\mathbb{P}(\Pi_L x_0 \in qM) = \mathbb{P}(q^* \Pi_L q x_0 \in M) = \mathbb{P}(\Pi_{q^*L} x_0 \in M) = \mathbb{P}(\Pi_L x_0 \in M), \tag{4}$$

since $L \sim q^* L$ due to the uniform distribution. If we now denote with u an element of $O(d-1)$ (which we will identify with the element of $O(d)$ which acts on $\mathrm{span}(e_d)^\perp$ as does u on $\mathbb{R}^{d-1}$, and leaves e_d invariant), we have

$$\Lambda(\rho, u\Theta) = \rho^2 x_0 + \rho\sqrt{1-\rho^2} Q_{x_0}[u\Theta, 0] = Q_{x_0} u Q_{x_0}^* \left(\rho^2 x_0 + \sqrt{1-\rho^2} Q_{x_0}[\Theta, 0]\right),$$

since $Q_{x_0} u Q_{x_0}^* x_0 = Q_{x_0} u e_d = Q_{x_0} e_d = x_0$. Therefore, we have

$$\begin{aligned}\mathbb{P}\left(R \in A, \theta \in uB\right) &= \mathbb{P}\left(\Pi_L x_0 \in \Lambda(A \times uB)\right) = \mathbb{P}\left(\Pi_L x_0 \in Q_{x_0} u Q_{x_0}^* \Lambda(A, B)\right) \\ &= \mathbb{P}\left(\Pi_L x_0 \in \Lambda(A \times B)\right) = \mathbb{P}\left(R \in A, \theta \in B\right),\end{aligned}$$

where we used the symmetry (4) and that $Q_{x_0} u^* Q_{x_0}^* x_0 = x_0$. Consequently, the left hand side of (3) is invariant under orthogonal transformations, i.e., uniformly distributed. This proves the claim. □

With the help of this lemma, we will be able to calculate the expected value of each entry of $\hat{x}$. In addition to this, we will use a concentration of measure argument – concretely, we will use the following result.

Lemma 3.2. *Let $k \leq d$. If $L \sim \mathcal{U}\left(G(d,k)\right)$ and $F : G(d,k) \to \mathbb{R}$ is 1-Lipschitz, i.e., for $K_1, K_2 \in G(d,k)$, we have*

$$|F(K_1) - F(K_2)| \leq \|\Pi_{K_1} - \Pi_{K_2}\|_2\,,$$

then there exist constants $C, a > 0$ such that, for every $t > 0$,

$$\mathbb{P}\left(F(L) \geq \mathbb{E}\left(F(L)\right) + t\right) \leq C\exp(-adt^2) \quad \text{and} \quad \mathbb{P}\left(F(L) \leq \mathbb{E}\left(F(L)\right) - t\right) \leq C\exp(-adt^2).$$

$\mathbb{E}\left(F(L)\right)$ is the expected value of the random variable $F(L)$.

Since the proof of this result is relatively lengthy and uses fairly standard techniques, we postpone it to Subsection 5.2. Let us instead apply it now to prove that the support approximation procedure (2) performs well as long as m is large enough.

Theorem 3.3. *Let $x_0 \in \mathbb{Z}^d$ be fixed, $S_0 = \operatorname{supp} x_0$, $\vartheta \in (0,1)$ with S_ϑ as in (2), A a Gaussian matrix, and $\eta > 0$. Further, let the solution of $(\mathcal{P}_2)$ be denoted $\hat{x}$ and $C, a > 0$ be the associated constants from Lemma (3.2). Then the following hold.*

(i) If $i \in S_0$ and

$$m \geq \sqrt{\frac{d\,\|x_0\|_2^2}{a\left(|x_0(i)| - \vartheta\right)^2} \cdot \log\left(\frac{C}{\eta}\right)},$$

then the probability that $i \notin S_\vartheta$ is smaller than η.

(ii) If $i \notin S_0$ and

$$m \geq \sqrt{\frac{d\,\|x_0\|_2^2}{\vartheta^2 a} \cdot \log\left(\frac{2C}{\eta}\right)},$$

then the probability that $i \in S_\vartheta$ is smaller than η.

Proof. For $i \in \{1, \dots, d\}$, let e_i be the i:th standard unit vector, and define the function

$$G_i : G(d,m) \to \mathbb{R}, \quad L \mapsto \frac{1}{\|x_0\|_2}\left\langle \Pi_L x_0, e_i \right\rangle.$$

Then G_i is 1-Lipschitz, since

$$|G_i(L) - G_i(K)| \leq \frac{1}{\|x_0\|_2}\|\Pi_L x_0 - \Pi_K x_0\|_2 \leq \|\Pi_L - \Pi_K\|_2\,.$$

Let us next only prove the first claim (i), since the other claim is proven analogously. We start by calculating the expected value of G_i as follows: Lemma 3.1 and the fact that $\ker A^\perp \sim \mathcal{U}\left(G(d,m)\right)$ imply that

$$G_i(\ker A^\perp) = \left\langle \Pi_{\ker A^\perp}\left(\frac{x_0}{\|x_0\|_2}\right), e_i \right\rangle = R^2\left\langle \frac{x_0}{\|x_0\|_2}, e_i \right\rangle + R\sqrt{1-R^2}\left\langle Q_{x_0/\|x_0\|_2}[\theta, 0], e_i \right\rangle,$$

where we adapted the notation of said lemma. Now, due to symmetry, we have

$$\mathbb{E}\left(\langle Q_{x_0/\|x_0\|_2}[\theta,0],e_i\rangle\right)=\mathbb{E}\left(\left\langle [\theta,0],Q^*_{x_0/\|x_0\|_2}e_i\right\rangle\right)=0.$$

It is furthermore well-known that $\mathbb{E}\left(R^2\right)=m/d$. Hence,

$$\mathbb{E}\left(G_i\right)=\frac{m}{d\,\|x_0\|_2}\left\langle x_0,e_i\right\rangle.$$

Now let us without loss of generality assume that $x_0(i)>0$. Then necessarily $x_0(i)\geq 1$, and $x_0(i)-\vartheta>0$. Using the concentration result Lemma 3.2, we can deduce that

$$\begin{aligned}\mathbb{P}\left(i\notin S_\vartheta\right)&=\mathbb{P}\left(|\hat{x}(i)|\leq\frac{m}{d}\cdot\vartheta\right)\leq\mathbb{P}\left(\|x_0\|_2\,G_i\leq\frac{m}{d}\cdot\vartheta\right)\\&=\mathbb{P}\left(G_i-\mathbb{E}\left(G_i\right)\leq\frac{m}{d\,\|x_0\|_2}\cdot(\vartheta-x_0(i))\right)\leq C\exp\left(-ad\left(\frac{m}{d\,\|x_0\|_2}\cdot(\vartheta-x_0(i))\right)^2\right)\leq\eta,\end{aligned}$$

provided m satisfies the assumption of the theorem. □

Note that the above result bounds the probability that an index is classified correctly by our proposed procedure *separately for each index.* It is, however, also possible to prove a more "global" statement. To formulate it, it is convenient to define

$$\|x\|_{-\infty}:=\min_{i:x(i)\neq 0}|x(i)|\quad\text{for each } x\in\mathbb{R}^d.$$

Please note that although the notation might indicate that $\|\cdot\|_{-\infty}$ is a norm, it is not even a quasi-norm.

Theorem 3.4. *Let $x_0\in\mathbb{Z}^d$ be supported on S_0, $A\in\mathbb{R}^{m,d}$ be Gaussian, and S_ϑ be defined as above. Let further n_1 and n_2 be positive integers, and $\tau>0$. There exist continuous functions ϑ_- and ϑ_+ defined on $\mathbb{R}_+\times\mathbb{N}\times\mathbb{N}$ and universal constants D and b such that, if $\vartheta_-(\tau,n_1,m)\leq\vartheta\leq\vartheta_+(\tau,n_2,m)$, we have*

$$\mathbb{P}\left(|S_\vartheta\cap S_0^c|\leq n_1\wedge|S_\vartheta^c\cap S_0|\leq n_2\right)>\left(1-\exp\left(-bm\tau^2\right)\right)\left(1-D\left(\binom{d-|S_0|}{n_1}+\binom{|S_0|}{n_2}\right)\exp\left(-bm\tau^2\right)\right).$$

We furthermore have

$$\vartheta_-(0,n_1,m)=\sqrt{\left(\frac{d}{m}-1\right)\frac{1}{d}},\quad\vartheta_+(0,n_2,m)=\|x_0\|_{-\infty}-\sqrt{\left(\frac{d}{m}-1\right)\frac{d}{d-1}}\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}\left(\sqrt{\log n_2}+\sqrt{\frac{2}{\pi}}\frac{\|x_0\|_\infty}{\|x_0\|_2}\right).$$

The functions ϑ_- and ϑ_+ are explicitly specified in the proof, which we postpone to Subsection 5.3. The idea of the proof is to use Lemma 3.1 and some standard high-dimensional geometry arguments.

3.1.2 Stability

In applications, it is of course crucial that the selection procedure is stable, in the sense that is not to any great extent affected by noise in the measurements. We now show that $PROMP$ is stable in this sense. The key to its proof will be the following lemma.

Lemma 3.5. *Let $A\in\mathbb{R}^{m,d}$ be a Gaussian matrix with $d=(1+\delta)m$ and $n\in\mathbb{R}^m$ be fixed. Further, let $\hat{\rho}$ be the solution of the problem*

$$\min\|\rho\|_2\ \textit{subject to}\ A\rho=r.$$

Then the distribution of $\hat{\rho}$ is rotational invariant. Furthermore, there exist c_1, c_2, $\tilde{c}_1$ and $\tilde{c}_2$ such that, if we have $\delta \geq \tilde{c}_1/\log(\tilde{c}_2 d)$, then

$$\mathbb{P}\left(\|\rho\|_2 \leq \frac{\|r\|_2}{c_1\sqrt{d}}\right) \geq 1 - \exp(-dc_2).$$

All constants are universal except for c_1 which is only dependent on δ.

Proof. In order to prove that the distribution of ρ is rotational invariant, take q to be an orthogonal matrix. Then $q\hat{\rho}(A) = \hat{\rho}(Aq^*)$. Since $A \sim Aq^*$, consequently $q\hat{\rho}(A) \sim \hat{\rho}(A)$, i.e., we have rotational invariance.

To show the second claim, it suffices to notice that $\rho \in \ker A^{\perp}$. This then implies that

$$\|r\|_2 = \|A\rho\|_2 \geq \sigma_{\min}(A)\,\|\rho\|_2\,,$$

and the claim follows immediately from [28, Theorem 3.1] about the distribution of the singular values of Gaussian matrices. □

Now we may easily prove stability.

Theorem 3.6. *Let $x_0 \in \mathbb{Z}^d$, let $A \in \mathbb{R}^{m,d}$ be Gaussian with $d = (1+\delta)m$, let $r \in \mathbb{R}^m$ satisfy $\|r\|_2 \leq \sigma\sqrt{m}$, let $\widehat{C} > 0$ be arbitrary, and let $\vartheta \in (\widehat{C}\sigma, 1 - \widehat{C}\sigma)$. Then there exist constants c_1, c_2, $\tilde{c}_1$ and $\tilde{c}_2$, C and a, all universal except for c_1 which only depends on δ, such that, if*

(i) $d \geq \max\{c_2/\log(6/\eta), (1+\delta)\sqrt{2}/(e\sqrt{\pi}\widehat{C}^2 c_1^2 \log(6/\eta))\}$ and

(ii) $\delta \geq \tilde{c}_1/\log(\tilde{c}_2 d)$,

we have, for $i \in S_0$,

$$m \geq \sqrt{\frac{d\,\|x_0\|_2^2}{a(|x_0(i)| - \vartheta - \widehat{C}\sigma)^2} \cdot \log\left(\frac{6C}{\eta}\right)} \Rightarrow \mathbb{P}\left(i \notin S_\vartheta\right) \leq \eta$$

and, for $i \notin S_0$,

$$m \geq \sqrt{\frac{d\,\|x_0\|_2^2}{a(\vartheta - \widehat{C}\sigma)^2} \cdot \log\left(\frac{12C}{\eta}\right)} \Rightarrow \mathbb{P}\left(i \in S_\vartheta\right) \leq \eta.$$

Proof. We only prove the first claim, since the second is proved similarly. For this, we first observe that, since the solution of $(\mathcal{P}_2)$ is linearly independent of b, we have

$$\hat{x} = \Pi_{\ker A^{\perp}} x_0 + \hat{\rho},$$

where $\hat{\rho}$ is the solution of $(\mathcal{P}_2)$ with $b = r$. We may proceed exactly as in the proof of Theorem 3.3 to show that if m satisfies the assumed bound, we have $\langle \Pi_{\ker A^{\perp}} x_0, e_i\rangle \leq m/d \cdot (\vartheta - \widehat{C}\sigma)$ with a probability less than $\eta/6$. Hence, if we prove that the probability that $|\langle \hat{\rho}, e_i\rangle| \geq \hat{C}\sigma$ is smaller than $5\eta/6$, the theorem follows immediately.

To show $|\langle \hat{\rho}, e_i\rangle| \geq \hat{C}\sigma$, first notice that Lemma 3.5 implies $\hat{\rho} = \lambda\theta$, where $\theta \sim \mathcal{U}\left(\mathbb{S}^{d-1}\right)$ and $\lambda \leq \|n\|_2/(c_1\sqrt{d})$ with a probability greater than $\exp(-c_2 d)$. A standard concentration of measure argument (see for example [3, Theorem 14.3]) shows that for every $1 > t > 0$, we have $|\langle \theta, e_i\rangle| \leq t$ with a probability greater than $1 - e\sqrt{\pi/2}\exp(-dt^2/2)$. Choosing $t = \sqrt{d/m}\hat{C}c_1$, we arrive at

$$|\langle \hat{\rho}, e_i\rangle| \leq \frac{t\sigma\sqrt{m}}{c_1\sqrt{d}} = \hat{C}\sigma$$

with a complimentary probability smaller than $\exp(-c_2 d) + e\sqrt{\pi/2}\exp(-d(1+\delta)\widehat{C}^2 c_1^2/2)$. The assumptions (i) and (ii) imply that this number is smaller than $5\eta/6$, completing the proof. □

Remark 3.7. *The careful reader might have already noticed that the arguments in this subsection only use the integer-valuedness of the signal x_0 in the sense that it secures that $\|x_0\|_{-\infty} \geq 1$. Hence, the technique of forming a support estimate with ℓ_2-minimization and thresholding will work just as well for any signal with this property.*

3.2 Analysis of Warm-Starting OMP Step

In this section, we aim to analyze the impact an initial support approximate has on the OMP algorithm. Let us first note that as soon as OMP has found a support approximation S which includes all the indices S_0, while at the same time having a size which is smaller than the number of measurements m, the minimization procedure $\min_{\operatorname{supp} x \subseteq S} \|Ax - b\|_2$ will have the unique solution x_0. This is due to the fact that the columns $(a_i)_{i \in S}$ of the Gaussian matrix A almost surely will be linearly independent. Hence, if we prove that the OMP-steps (line 6 to 8 in Algorithm 2) will pick indices $i \in S_0$, we will successfully recover our signal as long as the initial support estimate is not too large.

A famous criterion for OMP to successfully recover the support of an s-sparse signal involves the *mutual coherence* μ of the matrix A. If a_i, $i = 1, \dots, d$ denote the columns of the matrix, it is defined by[1]

$$\mu(A) := \sup_{i \neq j} |\langle a_i, a_j \rangle|$$

If A has normalized columns, a sufficient criterion for OMP to recover all s-sparse signals in s-iterations is that $\mu(2s - 1) < 1$ [18, p.123-124].

It is possible – with a proof technique similar to the result mentioned above – to show a corresponding criterion for the situation that OMP is initialized with a set S. For this, let us first define a new version of the mutual coherence.

Definition 3.8. *Let $A \in \mathbb{R}^{m,d}$ have the columns a_i, $i = 1, \dots, d$ and $S \subseteq [1, \dots, d]$. The* S-coherence *$\mu_S$ of A is defined by*

$$\mu_S(A) = \sup_{j \neq k} \left| \left\langle \Pi_{\operatorname{ran} A_S^\perp} a_j, \Pi_{\operatorname{ran} A_S^\perp} a_k \right\rangle \right|,$$

where A_S denotes the matrix formed by the columns $(a_i)_{i \in S}$.

Remark 3.9. *It is sufficient to consider $j, k \notin S$, since $a_j \in \operatorname{ran} A_S$ for $j \in S$ and, therefore, $\Pi_{\operatorname{ran} A_S^\perp} a_j = 0$.*

We may now formulate and prove a criterion for the success of the OMP-step in $PROMP$.

Theorem 3.10. *Let $x_0 \in \mathbb{R}^d$ have the support S_0 and $A \in \mathbb{R}^{m,d}$ have the columns a_i, $i = 1, \dots, d$. Denote the current support estimate of OMP with the measurements $b = Ax_0$ by S, and set*

$$I := \{ i \notin S : |x_0(i)| = \|x_0|_{S^c}\|_\infty \}.$$

Then the next step of OMP will pick an index in S_0, if

$$\max_{i \in I} \left\| \Pi_{\operatorname{ran} A_S^\perp} a_i \right\|_2^2 \geq \mu_S (2 |S_0 \backslash S| - 1). \tag{5}$$

Proof. Given the support estimate S, we know that the current signal estimate x satisfies $Ax = \Pi_{\operatorname{ran} A_S} b$. This means that the current residual ρ satisfies

$$\rho = b - Ax = \Pi_{\operatorname{ran} A_S^\perp} b = \sum_{j \in S_0} x_0(j) \Pi_{\operatorname{ran} A_S^\perp} a_j,$$

since $b = Ax_0$ and $x_0 = \sum_{j \in S_0} x_0(j) e_j$. Hence, using $\Pi_{\operatorname{ran} A_S^\perp} a_j = 0$ for $j \notin S$

$$\langle \rho, a_k \rangle = \sum_{j \in S_0} x_0(j) \left\langle \Pi_{\operatorname{ran} A_S^\perp} a_j, \Pi_{\operatorname{ran} A_S^\perp} a_k \right\rangle = \sum_{j \in S_0 \backslash S} x_0(j) \left\langle \Pi_{\operatorname{ran} A_S^\perp} a_j, \Pi_{\operatorname{ran} A_S^\perp} a_k \right\rangle.$$

[1] Some authors choose to normalize the columns. We however do not, since this approach will enable an analysis of the impact of the initialization with S.

Thus we have

$$\langle \rho, a_k \rangle = \begin{cases} 0 & : k \in S, \\ x_0(k) \left\| \Pi_{\operatorname{ran} A_S^\perp} a_k \right\|_2^2 + \sum_{j \in S_0 \setminus (S \cup \{k\})} x_0(j) \left\langle \Pi_{\operatorname{ran} A_S^\perp} a_j, \Pi_{\operatorname{ran} A_S^\perp} a_k \right\rangle & : k \in S_0 \setminus S, \\ \sum_{j \in S_0 \setminus S} x_0(j) \left\langle \Pi_{\operatorname{ran} A_S^\perp} a_j, \Pi_{\operatorname{ran} A_S^\perp} a_k \right\rangle & : k \notin (S \cup S_0). \end{cases}$$

Denoting $M := \max_{i \in I} |x_0(i)|$ and $i_0 := \operatorname{argmax}_{i \in I} |x_0(i)|$, this implies that

$$\begin{aligned} \left| \left\langle a_k, \Pi_{\operatorname{ran} A_S^\perp} b \right\rangle \right| &\geq M \left\| \Pi_{\operatorname{ran} A_S^\perp} a_k \right\|_2^2 - M(|S_0 \setminus S| - 1)\mu_S & : k = i_0, \\ \left| \left\langle a_k, \Pi_{\operatorname{ran} A_S^\perp} b \right\rangle \right| &\leq M \, |S_0 \setminus S| \, \mu_S & : k \notin I. \end{aligned}$$

Therefore, i_0 will be chosen prior to any index in I^c. In particular, since $I \subseteq S_0$, OMP will choose an index in the support of x_0, which we aimed to prove. □

Next we aim to investigate the probability that a Gaussian matrix $A \in \mathbb{R}^{m,d}$ fulfills 5, depending on S and S_0. For this, we require the following lemma, which is interesting in its own right.

Lemma 3.11. *Let $S \subseteq [1, \ldots, d]$ with $|S| \leq m$ be given, and let $A \in \mathbb{R}^{m,d}$ and $\tilde{A} \in \mathbb{R}^{m-|S|, d-|S|}$ be Gaussian matrices. Let further F be a measurable function on $\mathbb{R}^{m,d-|S|}$ which is left-invariant under orthogonal transformations, i.e., if $q \in O(m)$, then $F(qA) = F(A)$. Then*

$$F\left((\Pi_{\operatorname{ran} A_S^\perp} a_j)_{j \notin S} \right) \sim F(\tilde{A}),$$

where we identified $\tilde{A}$ with the matrix formed when concatenating it with $|S|$ zero rows.

Proof. Without loss of generality, we may assume that $S = [d - |S| + 1, \ldots, d]$, and therefore use the same indices for the vectors $\Pi_{\operatorname{ran} A_S^\perp} a_j$ and $\tilde{a}_\ell$. Because of the Gaussianity of A, $\operatorname{ran} A_S^\perp$ is uniformly distributed over the Grassmannian $G(m, m - |S|)$ of $(m - |S|)$-dimensional subspaces of $\mathbb{R}^m$, and also $\operatorname{ran} A_S^\perp \perp\!\!\!\perp (a_j)_{j \notin S}$, where we used the notation $\perp\!\!\!\perp$ to indicate that the two variables are independent. Therefore, we only have to show that given $\operatorname{ran} A_S^\perp$, the conditional distribution of $F\left((\Pi_{\operatorname{ran} A_S^\perp} a_j)_{j \notin S} \right)$ is equal to the one of $F(\tilde{A})$.

For this, fix $\operatorname{ran} A_S^\perp =: L \in G(m, k)$ (with $k = m - |S|$) and let q be an orthogonal matrix with $qL_k = L$, $L_k = \operatorname{span}(e_1, \ldots, e_k)$. Then, using the fact that, since a_j is Gaussian, $qa_j \sim a_j$ and $\Pi_{L_k} a_j \sim \tilde{a}_j$, and also identifying $\tilde{a}_j \in \mathbb{R}^{m-|S|}$ with the vector in $\mathbb{R}^m$ formed when concatenating $\tilde{a}_j$ with zeros, we have

$$\Pi_L a_j = q \Pi_{L_k} q^* a_j \sim q \Pi_{L_k} a_j \sim q \tilde{a}_j.$$

It is also trivial that the family $(\Pi_L a_j)_{j \notin S}$ is independent, as is $(q\tilde{a}_j)_{j \notin S}$. Hence,

$$(\Pi_L a_j)_{j \notin S} \sim (q\tilde{a}_j)_{j \notin S}.$$

The claim now follows from the left-invariance of F. □

Remark 3.12. *Some examples of entities of matrices $A \in \mathbb{R}^{m,d-|S|}$, which are left-invariant under orthogonal transformations, are*

- *their mutual incoherence, as well as the norms of their columns,*
- *their kernels,*
- *their singular values.*

With the help of the last lemma, it is now possible to investigate with which probability a Gaussian matrix A fulfills (5). Let us state the result.

Theorem 3.13. *Let $S \subseteq [1, \dots, d]$ with $|S| \leq m$ be given, and let $A \in \mathbb{R}^{m,d}$ a Gaussian matrix. Then there exist constants C, $a > 0$ such that, for every $\eta > 0$, the probability that A fulfills (5) is larger than $1 - \eta$, provided that*

$$m \geq |S| + \left(\frac{2\,|S_0 \backslash S|\,\sqrt{|I|} + 1}{2\sqrt{|I|}} \right)^2 \cdot \log \left(\frac{\sqrt{\frac{\pi}{2}}\tilde{d}(\tilde{d} - 1) + \tilde{d}}{\eta} \right),$$

where $\tilde{d} = d - |S|$.

The proof is quite technical and therefore postponed to Subsection 5.4. The main idea of it is as follows: By Lemma 3.11 and Remark 3.12, it suffices to bound the probability that

$$\max_{i \in I} \|\tilde{a}_i\|_2^2 \geq \mu(\tilde{A})(2\,|S_0 \backslash S| - 1), \tag{6}$$

where $\tilde{A} \in \mathbb{R}^{m-|S|,\tilde{d}}$ is Gaussian.

It is a standard measure concentration result that the norm of a Gaussian vector in $\mathbb{R}^n$ typically will have squared norm equal to n. Hence, the left hand side of (6) will usually be about $\tilde{m} = m - |S|$.

To take care of the right hand side, notice that

$$\mu(\tilde{A}) = \sup_{j \neq k} \|\tilde{a}_j\|_2 \|\tilde{a}_k\|_2 \left|\langle \theta_j, \theta_k \rangle\right|,$$

where $(\theta_j) = (\tilde{a}_j / \|\tilde{a}_j\|_2)$ is an independent family of vectors, each one uniformly distributed over the sphere. The product $\|\tilde{a}_j\|_2 \|\tilde{a}_k\|_2$ will again typically be smaller than $\tilde{m}$, and each scalar product $|\langle \theta_j, \theta_k \rangle|$ will – again according to classical measure of concentration results – with high probability be smaller than $1/\sqrt{\tilde{m}}$. The fact that these bounds need to hold for $\tilde{d}$ vectors and $\tilde{d}(\tilde{d} - 1)/2$ scalar products causes an extra multiplicative $\log(\tilde{d})$-term. Summarizing, $\mu(\tilde{A}) \approx \tilde{m} \log(\tilde{d})/\sqrt{\tilde{m}}$, and (6) is hence fulfilled when

$$\begin{aligned} \tilde{m} &\geq \tilde{m} \cdot \frac{\log(\tilde{d})}{\sqrt{\tilde{m}}} \cdot (2\,|S_0 \backslash S| - 1), \text{ i.e.} \\ m &\geq |S| + (2\,|S_0 \backslash S| - 1)^2 \log^2(\tilde{d}), \end{aligned}$$

which almost is the claim of the theorem. A more thorough analysis will produce the actual result.

Remark 3.14. *Let us already now remark that these considerations give a theoretical argument why a warm start should enhance the recovery probability. Taking $S = \emptyset$ (for which the theorem still is valid), it follows that we require $O(|S_0|^2 \log(d))$ measurements to secure the recovery probability of a signal supported on S_0 using OMP. The square-term is not surprising, and is related to the so called square-root bottleneck [30]. With the initial support estimate S, we will instead need $O(|S| + |S \backslash S_0|^2 \log(d))$ measurements to secure recovery. This expression grows significantly provided $S \approx S_0$.*

It is however possible to prove a stronger recovery result. In [43], the authors prove that it actually suffices to use $m \gtrsim C\,|S_0| \log(d)$ measurements in order to with great probability recover a signal supported on S_0 with the help of OMP. This result can be generalized to our setting in a similar manner.

Theorem 3.15. *Let $S \subseteq [1, \dots, d]$ with $|S| \leq m - |S_0 \backslash S|$ be fixed, and let $A \in \mathbb{R}^{m,d}$ be Gaussian. Let further $x_0 \in \mathbb{Z}^d$ be supported on the set S_0 and $\eta > 0$. Then there exists a constant K such that the probability that OMP warm-started with the index set S will recover x_0 from the measurements $b = Ax_0$ is larger than $1 - \eta$ provided*

$$m \geq |S| + K\,|S \backslash S_0| \log \left(\frac{d - |S|}{\eta} \right).$$

Moreover, K can always be chosen smaller than 16.

The idea of this proof (which is from [43]) is less intuitive than the one of the above result, and we choose to fully postpone the presentation of it to Subsection 5.5.

4 Numerical Experiments

In this section, we empirically investigate the performance of $PROMP$ and OMP. We will begin by comparing the recovery probabilities of OMP and $PROMP$ when they are fed with exact measurements. Then we will investigate the effect of noise on the performance of $PROMP$.

In the spirit of reproducible research, we have made the code used in the experiments available as an open source MATLAB software package. It can be downloaded from `https://www.math.tu-berlin.de/afg/`.

4.1 Exact Recovery

We start by numerically testing the recovery probability for $PROMP$ and OMP without noise. The experiment was conducted as follows: For each sparsity level s and number of measurements m from 1 to 100 (= ambient dimension), we ran 1000 experiments. In each of those, we drew a support of size s uniformly at random, and chose the elements on that support equal to 1 to form the vector x_0 (i.e., we are working with vectors in $\{0,1\}^d$). Then we calculated $b = Ax_0$ and ran both $PROMP$ and OMP to estimate x_0. Since the elements of the original vectors are all positive, we also tested the recovery probability, if the elements to form the support approximation were chosen as the ones which were larger than $d/m \cdot 1/2$ (as opposed to larger than $d/m \cdot 1/2$ in modulus). A success was declared when the Euclidean distance between the estimate x and the true signal x_0 was at most 0.001 (which corresponds to a relative error between $0.1 - 1‰$, depending on the sparsity).

In the left hand diagrams of Figure 4, the success probability of $PROMP$ (including the successes already in the preconditioning step) to OMP are compared. As can be seen, the phase transition region is pushed downwards by the ℓ_2-preconditioning, in particular, in the case when the sparsity is not too small. When the index classification procedure specialized for positive signals is used, the phase transition region is pushed even further down.

Another gain when using $PROMP$ instead of OMP is the reduction of required computation time. In the right hand diagrams of Figure 4, the average time needed for a successful execution is plotted for OMP as well as for $PROMP$ (with the original index classification procedure), respectively. Certainly, a non-successful execution already requires extensive computational time, since unless a sparse solution is found at an early stage, many iterations are necessary to find a solution at all. One should also point out that there is a larger statistical instability on the borderline to the white area in which no attempts actually succeeded; on the borderline only a few attempts succeeded and hence only a few execution times were recorded. We further wish to emphasize that since we have chosen the same scale (ranging from 0 to 0.001 seconds) for the figures, the vast superiority of $PROMP$ compared to OMP for large sparsity levels can not be captured by them. As an example, a successful attempt of $PROMP$ for $s = 70$, $m = 95$ was averagely taking 0.024 seconds, whereas $PROMP$ (with and without positivity specialized support) only needed an average of 0.0012 seconds. It should also be mentioned that no efforts whatsoever where made in order to increase the speed of OMP, e.g., by using QR-decompositions to solve the ℓ_2-minimization problem, but instead literally does what the pseudo code Algorithm 1 says.

Since the introduction of OMP, many modifications of this sparse recovery algorithm have been presented. In order to incorporate $PROMP$ in this larger context, we chose to also present a comparison with two of these modifications, namely $CoSaMP$ [32] and $OMPR$ (Orthogonal Matching Pursuit with Replacement) [22]. For each of the two algorithms, we ran experiments similar to the ones above, and checked for each sparsity level how many measurements were needed to secure a recovery probability above 90%.

We then tested to which extent the $CoSaMP$ and $OMPR$ can be improved by using the Support Approximation Step of $PROMP$. For this, we performed the same test for "pre-projected" versions of the two algorithms, i.e., we warm-started each algorithm with the set S_ϑ, again with $\vartheta = \frac{1}{2}$. The resulting algorithms are called $CoSaPROMP$ and $PROMPR$ below.

The results of the experiments are shown in Figure 5. One can see that with the exception of $CoSaMP$ for sparsities up to about 35, $PROMP$ performs comparably good or even better than all of the other algorithms. We also see that the pre-projection procedure enhances the recovery probabilities of both $OMPR$ and $CoSaMP$.

The curves corresponding to $CoSaMP$ and $CoSaPROMP$ make a sudden jump downwards at a sparsity of about $s = 25$. As can be seen in Figure 6, there seems to be a "sweet spot" when the number of measurements m

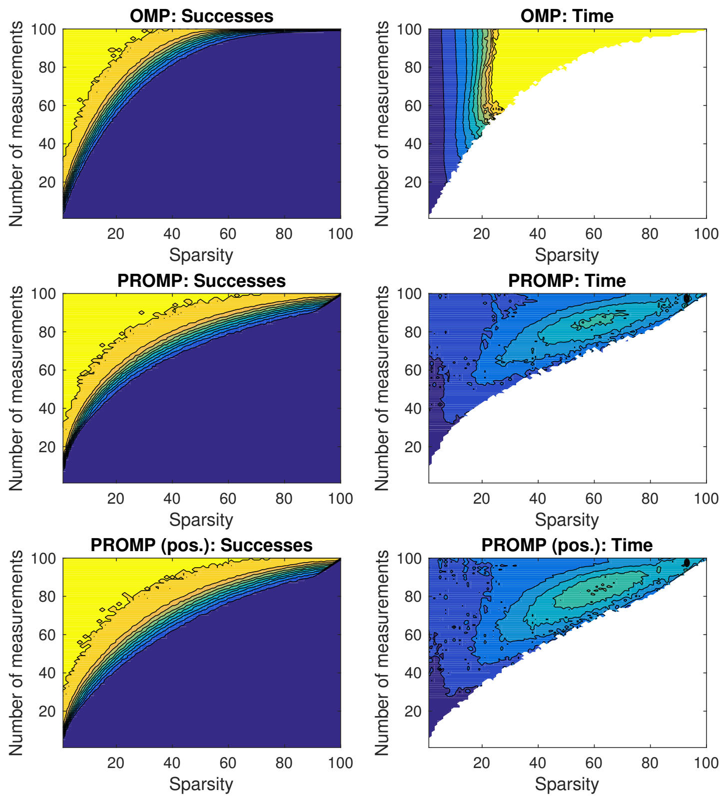


Figure 4: Left, from top: Recovery probabilities OMP, $PROMP$ without and and with positivity specialized support. Right, from top: Average time needed to successfully execute OMP, $PROMP$ without and and with positivity specialized support. Scale ranges from 0 to $3 \cdot 10^{-3}$ seconds.

is slightly larger than s, where $CoSaMP$ performs better than when m is both larger and smaller than this value. In our opinion, there seem to not be a simple explanation of this effect, which could be an interesting question for future research.

Let us end with a few more comments about the way $CoSaMP$ and $OMPR$ were applied:

- Both algorithms need knowledge of the sparsity of the signal, which we provided exactly (i.e., in the experiments with s-sparse signals, we gave the algorithm exactly s as sparsity parameter, except for $CoSaMP$ in some cases, see below.).
- The $CoSaMP$-algorithm consists of iteratively constructing $2s$-sparse signals, which are then thresholded to s-sparse signals. This makes it unclear how it should be applied to signals with sparsity greater than $\frac{d}{2}$. We chose to simply set the sparsity equal to $\frac{d}{2}$ in this case.
- $OMPR$ needs to be initialized with an s-sparse signal. We chose this signal to be the one consisting of the s largest entries of the least squares solution to $Ax = b$.

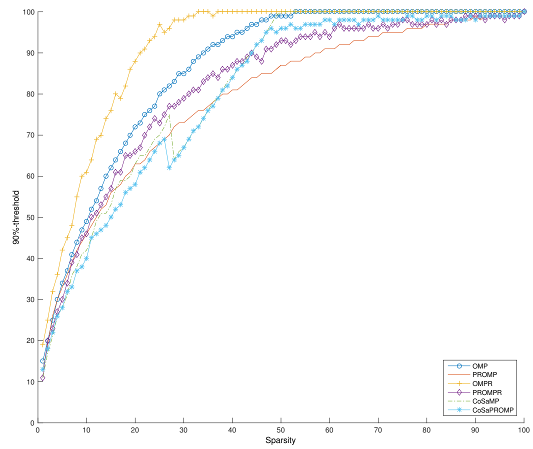


Figure 5: Curves representing the number of measurements needed to obtain a recovery probability of 90% for each sparsity for different algorithms. .

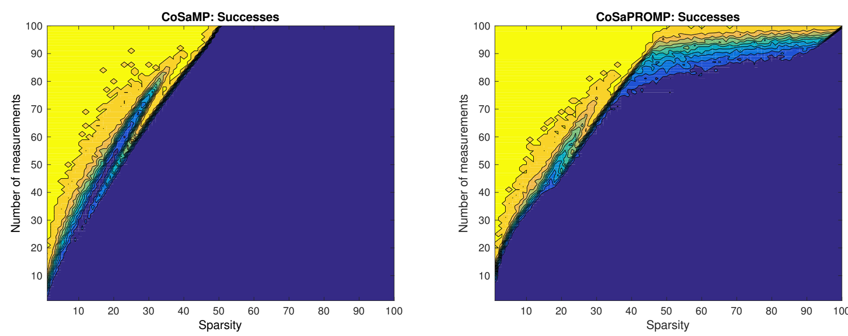


Figure 6: The behaviour of CoSaMP and CoSaPROMP.

4.2 Stable Recovery

We first study the effect of noise on the Support Approximation Step of $PROMP$. For the experiment, we chose the ambient dimension to be equal to $d = 100$ and the sparsity to be $s = 10$. For each $m = 10, 20, \dots, 90$ as well as $m = 91, 92 \dots 100$, a signal x_0 was chosen in the same manner as before. For each m, we let $PROMP$ run 10000 times, one time without noise and one time with noise. The noise was chosen uniformly at random on the sphere $\mathbb{S}^{m-1}$ with radius $0.1 \cdot \|b\|_2$, i.e., corresponding to a relative noise level of 10%. For each experiment, the sizes of $S \cap S_0$ and $S \cap S_0^c$, i.e., the number of chosen indices which were correct and false, respectively, were recorded. The results are depicted in Figure 7.

We observe that as long as m is not too large, a relative noise level of 10% is really not much of an issue for $PROMP$ in this situation, both with and without the positivity specialized index choice procedure – the number of correctly, and incorrectly, respectively, chosen indices are not affected to any great extent. If however $m \approx d$, we really do experience problems, as was anticipated by Theorem 3.6. We also find an explanation why $PROMP$ is not outperforming OMP for small sparsities. The theorems in Subsection 3.2 concerning the recovery probability of OMP initialized with a set S make the assumption that S is not too large – the figure shows that this assumption typically is not met for small sparsities and the threshold $1/2$.

Next, we investigate how different noise levels affect the performance of OMP and $PROMP$, respectively.

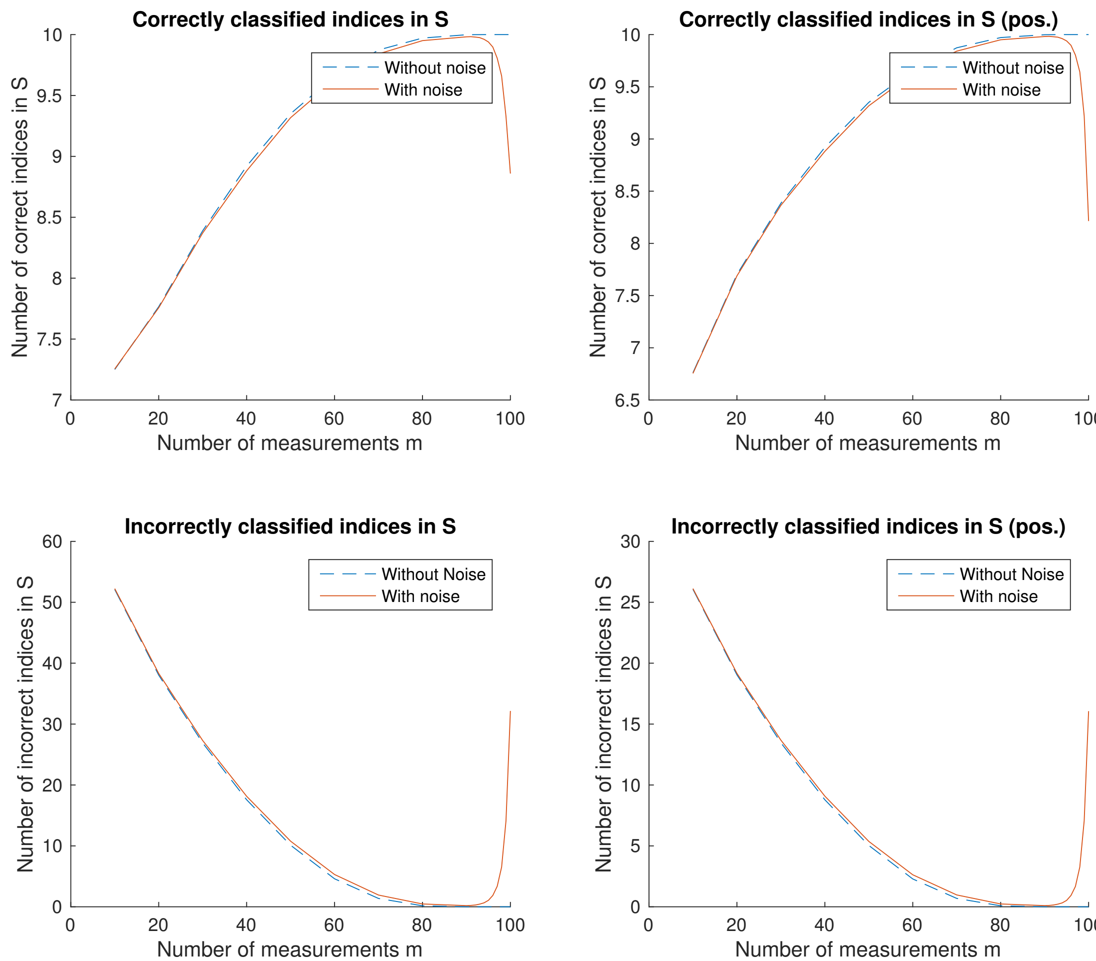
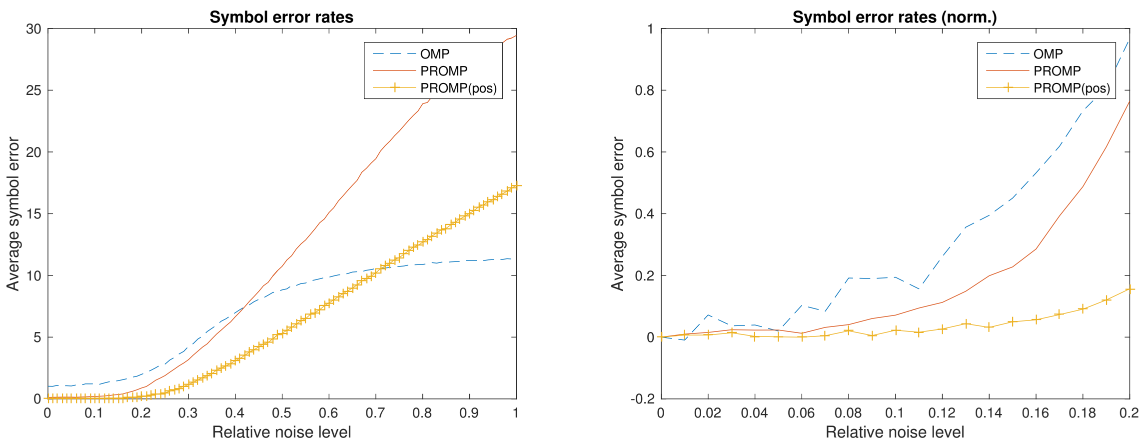


Figure 7: Top row, left to right: $|S \cap S_0|$ depending on m and the presence of noise for the normal index classification procedure and for the positivity specialized procedure. Bottom row, left to right $|S \cap S_0^c|$ depending on m and the presence of noise for the normal index classification procedure and for the positivity specialized procedure.

Here we fixed $d = 100$, $s = 10$ and $m = 50$.For these values of m and s, when recovering from exact measurements, OMP has a recovery probability of about 92%, $PROMP$ about 98%, and $PROMP$ with positivity specialized support about 99%, respectively, according to the experiments in the last section. The signal x_0 and the support estimate S was chosen as before. Then for each relative noise level $0, 0.01, 0.02, \ldots, 1$, we performed 10000 OMP's and $PROMP$'s to recover x_0. We chose to terminate the orthogonal matching pursuits, both in $PROMP$ and in OMP, when $\|Ax - b\|_2$ dropped below $\|n\|_2$, or after at most 15 iterations. After the algorithms had ran, we rounded off (post-projected) the answers and recorded how many entries in the solution vector $\hat{x}$ were different from the ones in x_0. The mean number of false entries depending on the noise level is depicted in Figure 8(a).

It is evident from the figures that $PROMP$, especially with the positivity specialized index choice procedure, outperforms OMP by a large margin, at least for small noise levels. The reason of why OMP starts to work better for large noise levels is our choice of termination criterion for orthogonal matching pursuits, in OMP as well as in $PROMP$. Since the OMP-step in $PROMP$ is started with a support estimate already containing some false indices (in this setting, typically around 15-20 for the standard , and 5-10 for the positivity specialized index classification proceudere, respectively, see Figure 7), it will have the chance to choose more false indices before it is artificially stopped compared to the original OMP. We also remark that for noise levels up to about .06, or even .16 for the positivity specialized support, the noise is not affecting the performance significantly at all. This is due to the fact that we are post-projecting.



(a) (b)

Figure 8: (a): The mean number of false indices depending on relative noise level for OMP and $PROMP$, with and without positivity specialized index choice procedure. (b): The difference between the mean value at each level of relative noise and the mean value without noise for OMP and $PROMP$, with and without positivity specialized index choice procedure.

One issue to be taken into account is that OMP and $PROMP$ have different recovery probabilities from exact measurement for these values of s and m, which is the main reason why the mean value of false entries is larger for OMP. But also when we plot the difference between the mean value at each noise levels to the mean value at $SNR = 0$ for $SNR = 0, 0.01, \dots, 0.2$ in Figure 8(b), we notice that both variants of $PROMP$ also in this sense perform better than OMP.

5 Proofs

5.1 Proof of Theorem 2.2

The key to proving Theorem 2.2 is [13, Thm. 1], which states that for signals with positive entries, uniform recovery of all k-sparse signals in $\mathbb{R}^d$ is equivalent to the image of the d-simplex $P = \{x | x \geq 0, \|x\| = 1\}$ under the linear map A being a *k-neighborly* polytope with d vertices. Recall that a k-neighborly polytope is defined to be a polytope for which the convex hull of any set of k of its vertices forms a $(k-1)$-face of the polytope. A more simple way to phrase this equivalent condition is that each $(k-1)$-face of P should be mapped onto a $(k-1)$-face of AP.

The authors also briefly mention that the same reasoning can be applied to the ℓ_1 unit ball (cross-polytope) T^d when dealing with recovery of general sparse signals. It is furthermore possible to state and prove, again using the same techniques as in the mentioned paper, a local version of the theorem, which we will need. For completeness purposes, we include a proof. We will use some basic well-known results from convex geometry, which can be found in, e.g., [21].

Lemma 5.1. *Consider the problem $(\mathcal{P}_1)$ with $G = \mathrm{id}$ and $b = Ax_0$, where $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$ and $A \in \mathbb{R}^{m,d}$.*

(i) If $(\mathcal{P}_1)$ has the unique solution x_0, then $x_0/\left\|x_0\right\|_1$ lies in the relative interior of an s-face F of T^d which is mapped by A onto an s-face AF of AT^d.

(ii) If $x_0/\left\|x_0\right\|_1$ lies in the relative interior of an s-face F of T^d which is mapped by A onto an s-face AF of AT^d, and AF has the additional property that only the columns a_j of A contained in AF are the images of the vertices of F under A, then x_0 is the unique solution of $(\mathcal{P}_1)$.

In particular, if x_0 is a unique solution to $(\mathcal{P}_1)$, x_0 has to be m-sparse.

Proof. Let us again stress that the idea of this proof is taken from [13, Thm. 1]. For the sake of brevity, define the polytope $Q := AT^d$. We will subsequently without loss of generality assume that $\left\|x_0\right\|_1 = 1$. Consequently, $x_0 \in \operatorname{relint} F$ for some s-face F of T^d, i.e., there exist subsets I^+, I^- of $[1, \ldots, d]$ with $|I^+ \cup I^-| = s+1$ as well as positive scalars θ_i with $\sum_{i=1}^d \theta_i = 1$ and $\theta_i > 0$ for all $i \in I^+ \cup I^-$ such that

$$x_0 = \sum_{i \in I^+} \theta_i e_i - \sum_{i \in I^-} \theta_i e_i.$$

(i) If x_0 is the unique solution to $(\mathcal{P}_1)$, then

$$b = \sum_{i \in I^+} \theta_i a_i - \sum_{i \in I^-} \theta_i a_i \tag{7}$$

is the only way of writing b as a convex combination of the columns of A (the coefficients of any other convex combination corresponds to another solution of $(\mathcal{P}_1)$, which contradicts the assumption that x_0 uniquely solves $\mathcal{P}_1$). This immediately implies that the set $(a_i)_{i \in I^+} \cup (-a_i)_{i \in I^-}$ is affinely independent.

We claim that this also implies that $\operatorname{conv}((a_i)_{i \in I^+} \cup (-a_i)_{i \in I^-})$ is a subset of the relative boundary of Q. Towards a contradiction, assume this is not true. Then there would exist some $\tilde{b} \in AF \cap \operatorname{relint} Q$, consequently with two representations as convex combinations of the a_i:

$$\tilde{b} = \sum_{i \in \tilde{I}^+} \tilde{\theta}_i a_i - \sum_{i \in \tilde{I}^-} \tilde{\theta}_i a_i = \sum_{j=1}^d \vartheta_j^+ a_j - \sum_{j=1}^d \vartheta_j^- a_j$$

with $\tilde{I}^\pm \subseteq I^\pm$ and where both the $\tilde{\theta}_i$ and $\vartheta_i^\pm$ sum up to 1. Furthermore, all indices $\tilde{\theta}_i$, $i \in \tilde{I}^+ \cup \tilde{I}^-$ and $\vartheta_j^\pm$, $j = 1, \ldots d$ are strictly positive. This yields an affine combination of the a_i equal to 0, where the coefficents corresponding to indices $i \notin I^+ \cup I^-$ are positive. Adding a small multiple of this combination to (7) again yields an alternative way of expressing b as a convex combination of the a_i, which is a contradiction.

Since $AF = \operatorname{conv}((a_i)_{i \in I^+} \cup (-a_i)_{i \in I^-})$ is contained in the boundary, there exists a s-face G of Q with $AF \subseteq G$. A dimensionality argument shows that for every vertex v of G, there exists an $a_i, i \in I^+ \cup I^-$ whose representation as a convex combination of the vertices of G have a positive weight on v. If this v does not equal a_i, this convex combination can be used, just as above, to produce an alternative representation of b. Therefore, the vertices of AF and G are the same, i.e., $AF = G$. The theorem is proved.

(ii) The fact that $AF = \operatorname{conv}((a_i)_{i \in I^+} \cup (-a_i)_{I^-})$ is an s-face of Q immediately implies that the set $(a_i)_{i \in I^+} \cup (-a_i)_{I^-}$ is affinely independent – otherwise, the face could not have been s-dimensional. Now assume, towards a contradiction, that x_0 is not the unique solution of $(\mathcal{P}_1)$. Then there exists a tuple of nonnegative scalars $(\vartheta_j)_j$ not equal to $(\theta_i)_i$ and disjoint subsets J^+ and J^- of $[1, \ldots, d]$ such that

$$b = \sum_{j \in J^+} \vartheta_j a_j - \sum_{j \in J^-} \vartheta_j a_j, \tag{8}$$

where $\sum_{j=1}^d \vartheta_j \leq 1$ and $\vartheta_j > 0$ for all $j \in J^+ \cup J^-$.

Now, since AF is a face of Q, there exists a functional c and a scalar γ with

$$\langle c, x \rangle \leq \gamma \text{ for all } x \in Q \quad \text{and} \quad \langle c, x \rangle = \gamma \text{ for all } x \in AF.$$

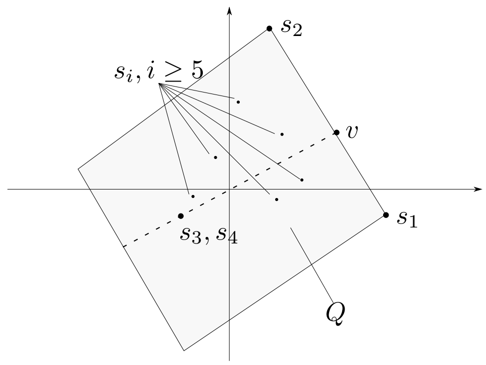


Figure 9: The construction of the matrix S in the proof Theorem 2.2(ii).

Since $b \in AF$, we have $\langle c, b \rangle = \gamma$. Testing equation (8) with c implies that $\langle c, \pm a_j \rangle = \gamma$ for each $j \in J^{\pm}$ as well as $\sum_{j \in J^+ \cup J^-} \vartheta_j = 1$. By the additional assumption regarding the columns of A, we obtain $\langle c, \pm a_k \rangle < \gamma$ for all $k \notin J^+ \cup J^-$, and hence $I^{\pm} = J^{\pm}$. Since the set $(a_i)_{i \in I^+} \cup (-a_i)_{I^-}$ is affinely independent, this implies that $(\vartheta_j)_j$ can not be disjoint from $(\theta_i)_i$ after all, a contradiction. Hence, x_0 is the unique solution of $(\mathcal{P}_1)$.

To prove the "in-particular"-part, we use (i) to conclude that any solution $\hat{x}$ of $(\mathcal{P}_1)$ must lie in an s-face $(e_i)_{i \in I^+} \cup (-e_i)_{I^-}$ with $I^+ \cap I^- = \emptyset$ of T^d, which is mapped to an s-face of Q. Since $Q \subseteq \mathbb{R}^m$, all faces of Q have dimension at most m, and hence $s \leq m$. To rule out the possibility $s = m$ (which corresponds to $\hat{x}$ being $(m+1)$-sparse), note that in this case, $AF = Q$. In particular, $(a_i)_{i \in I^+} \cup (-a_i)_{I^-}$ would be exactly the vertices of Q. This is however possible, since if a_i is a vertex of Q, $-a_i$ will also be one by symmetry. Hence I^+ and I^- cannot be disjoint, which is a contradiction. □

With this lemma, we may now prove the theorem.

Proof of Theorem 2.2. (i) Let $x_0 \in G\mathbb{Z}^d$ be $s - G$-sparse and $\hat{v}$ be the unique solution of $(\mathcal{P}_1)$ with $b = Ax_0$. Assume that $\hat{v} \neq v_0$, but still $\hat{v} \in v_0 + \Omega_P$. By Lemma 5.1, $\hat{v}$ has to be m-sparse. Therefore, if the length of the intersection of $\operatorname{supp} \hat{v}$ and $\operatorname{supp} v_0$ equals k, we can conclude that $v_0 - \hat{v}$ is a vector of the form described by (a) and (b) with $\ell \leq s - k$. To be precise, using the notation of the theorem, we have

$$v(i) = \begin{cases} v_0(i) - \hat{v}(i) & : i \in \operatorname{supp} \hat{v}, \\ 0 & : \text{else}, \end{cases} \quad \text{and} \quad n(i) = \begin{cases} v_0(i) & : i \in \operatorname{supp} v_0 \setminus \operatorname{supp} \hat{v}, \\ 0 & : \text{else}. \end{cases}$$

It is only left to prove that it is sufficient to consider only $\ell > 0$. For this, suppose that there exist m-sparse vectors in Ω_P, but none of them can be decomposed as described by (a) and (b). This, in particular, implies that there do not exist any vectors with ℓ_∞-norm greater or equal to 1 in Ω_P – if there were, by scaling, (remember that Ω_P is a convex polytope containing the origin) we would find a vector with ℓ_∞-norm equal to 1, i.e., a vector containing a 1, which would then be a vector of the form given by (a) and (b).

However, if there do not exist any vectors with ℓ_∞-norm greater or equal to 1 in Ω_P, already Theorem 2.1 tells us that there can not exist an x_0 as described above.

(ii) Let $v + n \in \Omega_P$, where v and n fulfill (a) and (b), and assume without loss of generality that $\operatorname{supp} v = [1, \ldots, m]$ and $\operatorname{supp} n = [m+1, \ldots, m+\ell]$. We now construct the matrix $S = AG$ as follows, and then just define A to be SG^{-1}.

First, we choose the m first columns s_i of S to be linearly independent vectors such that $(\pm s_i)$ are the vertices of a centrally $(m-1)$-neighborly polytope Q, meaning that each set of m vectors $\pm s_i$ not containing an antipodal pair spans an $m-1$-face of Q. One possible choice is to set $s_i = e_i$ for each i.

Next define $s := Sv$ and put $s_j := -s/(\ell n(j))$ for $j = m+1, \ldots, m+\ell$. We claim that each such column can not be an element of a face F of $\|v\|_1 Q$. To see this, let a and α be such that $\langle a, x \rangle \leq \alpha$ for all $x \in \|v\|_1 \cdot Q$ and

$\langle a, x\rangle = \alpha$ for all $x \in F$. Due to the linear independence of the s_i, $i = 1, \dots, m$, $0 \notin F$ and hence $\alpha \neq 0$. Since $\langle a, s\rangle \leq \alpha$, we have

$$\left|\left\langle a, -\frac{s}{\ell n(j)}\right\rangle\right| \leq \frac{1}{2}\left|\langle a, s\rangle\right| \leq \frac{|\alpha|}{2} < \alpha,$$

since $\ell \geq 2$ and $|n(j)| \geq 1$ for each j, and thus $-s/(\ell n(j)) \notin F$ for all $j = m+1, \dots m+\ell$.

Finally, fill the rest of the columns of S with vectors in the interior of Q, which then ensures the following conditions to be satisfied:

- $v/\left\|v\right\|_1$ lies in the relative interior of an k-face F of T^d which is mapped onto a k-face SF of $Q = ST^d$ by S (due to the centrally m-neighborliness of Q).
- There do not exist any columns in S other than the images of the vertices of F under S contained in the face SF.
- $b := s = Sv = -Sn$ is an element of that face.

Lemma 5.1(ii) therefore implies that the unique solution of $(\mathcal{P}_{1,G})$ with $b = Sv = -Sn$ has the unique solution v, which is not equal to $-n$, but still $v + n \in \Omega_P$. This concludes the proof. □

5.2 Proof of Lemma 3.2

In this subsection, we aim to prove Lemma 3.2. We will mainly use classical results on measure concentration in metric measure spaces, as presented in the monograph [27]. Let us begin with some useful definitions.

Definition 5.2. *[27, p.3] A metric space (X, d) equipped with a probability measure μ defined on its Borel σ-algebra $\mathcal{B}(X)$ is called a* metric measure space. *Given such a space, the* concentration function $\alpha_{(X,d,\mu)}$ *is defined by*

$$\alpha_{(X,d,\mu)}(r) = \sup\left\{1 - \mu(A_r) \mid \mu(A) \geq \frac{1}{2}\right\},$$

where for any set $A \subseteq X$ and $r > 0$, A_r denotes the set $\{x \in X \,|\, \mathrm{dist}(x, A) < r\}$.

If the concentration function of a metric measure space (X, d, μ) is rapidly decaying, 1-Lipschitz functions will be concentrated around their means, i.e., the probability that $|F - \mathbb{E}(F)| > \epsilon$ will be small. The exact statement reads as follows:

Theorem 5.3. *[27, p.10] Let X be a metric measure space with concentration function α. Suppose that $\alpha(r) \leq C\exp(-ar^p)$ for some C, a and $p > 0$. Then there exist C', $a' > 0$ such that for each 1-Lipschitz function F, we have*

$$\mu\left(F \geq \mathbb{E}(F) + r\right) \leq C'\exp(-a'r^p)\ ,\ \mu\left(F \leq \mathbb{E}(F) - r\right) \leq C'\exp(-a'r^p).$$

The metric measure space which is most relevant for this subsection is the Grassmannian $G(d,k)$ equipped with the metric $d(L, K) = \|\Pi_L - \Pi_K\|$ and the uniform probability measure $\mu_{G(d,k)}$. This measure $\mu_{G(d,k)}$ can be defined with the help of the fact that the orthogonal group $O(d)$ is acting on $G(d,k)$. If θ denotes the normalized Haar measure on $O(d)$ and L a fixed element of $G(d,k)$, one defines [25, p.93]

$$\mu_{G(d,k)}(B) := \theta\left(\{\theta \mid \theta L \in B\}\right).$$

By Theorem 5.3, proving $\alpha_{G(d,K)} \leq C\exp(-ar^2)$ for some $C, a > 0$ immediately implies Lemma 3.2. Our strategy will be to use a similar result on the sphere $\mathbb{S}^{d-1}$.

Let us first state and present a proof of this relatively well-known result.

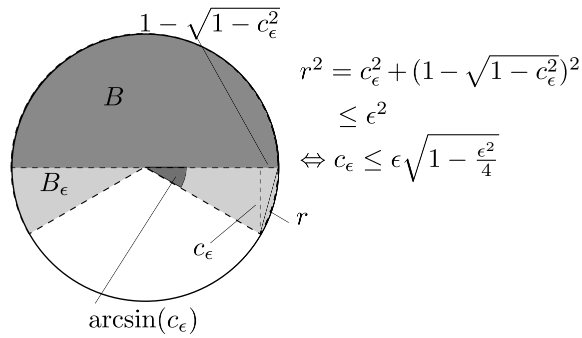


Figure 10: $c_\epsilon = \epsilon\sqrt{1-\frac{\epsilon^2}{4}}$.

Lemma 5.4. *We have*

$$\alpha_{\mathbb{S}^{d-1}}(r) \leq \sqrt{\frac{\pi}{8}}\exp(-dr^2/4), \tag{9}$$

where $\mathbb{S}^{d-1}$ is equipped with the normalized uniform measure σ^{d-1} and the metric inherited from $\mathbb{R}^d$.

Proof. Let A be a subset of $\mathbb{S}^{d-1}$ with $\sigma^{d-1}(A) = 1/2$, and B a *spherical cap* with radius $\pi/2$, i.e., a set of the form

$$B = \left\{x \in \mathbb{S}^{d-1} | \operatorname{dist}_{\mathbb{S}}(x, a) \leq \frac{\pi}{2}\right\},$$

where $\operatorname{dist}_{\mathbb{S}}$ is the distance with respect to the geodesic metric $d_{\mathbb{S}}(x,y) = \arccos(\langle x, y\rangle)$. The famous *isoperometric inequality* then implies that

$$1 - \sigma^{d-1}(A_\epsilon) \leq 1 - \sigma^{d-1}(B_\epsilon).$$

A proof of this very deep result can be found in, for instance, [16]. Thus it suffices to upper bound $1-\sigma^{d-1}(B_\epsilon)$.

For this, we observe that $\mathbb{S}^{d-1}\backslash B_\epsilon$ is a spherical cap C_ϵ of radius $\pi/2 - \arcsin(c_\epsilon)$, where $c_\epsilon = \epsilon\sqrt{1-\frac{\epsilon^2}{4}}$ (see Figure 10 for an illustration). Now, the measure of C_ϵ can be calculated by elementary integration, and we use [3, p.58], which proves that

$$\sigma^{d-1}(C_\epsilon) \leq \sqrt{\frac{\pi}{8}}\exp(-\arcsin(c_\epsilon)^2(d-1)/2).$$

Since $\epsilon \leq \sqrt{2}$ (C_ϵ is non-trivial only for these values of ϵ), this implies

$$\arcsin(c_\epsilon)^2 \geq c_\epsilon^2 = \epsilon^2(1-\epsilon^2/4) \geq \epsilon^2/2.$$

Hence we may estimate $\exp(-\arcsin(c_\epsilon)^2(d-1)/2) \leq e\exp(-d\epsilon^2/4)$, where we again used $\epsilon \leq \sqrt{2}$. This proves the claim. □

The reason why it is possible to deduce a rapid decay for the concentration function on $G(d,k)$ from the rapid decay of the one of $\mathbb{S}^{d-1}$ is the fact that the uniform probability measures on the two spaces are related as follows: if we fix $L \in G(d,k)$ and $\nu \in \mathbb{S}^{d-1}$, for each Borel set $B \subseteq G(d,k)$, we have

$$\mu_{G(d,k)}(B) = \theta\left(\{\theta \mid \theta L \in B\}\right) = \sigma^{d-1}\left(\{q\eta \mid qL \in B\}\right). \tag{10}$$

The first equality is simply the definition of the measure $\mu_{G(d,k)}$ and the second is proven in [25, p.91]. With this formula, we may deduce the rapid decay of $\alpha_{G(d,K)}$. For this, we first require the following lemma.

Lemma 5.5. *Let $r > 0$ and $\eta,\ \rho \in \mathbb{S}^{n-1}$. Then there exists some $\hat{q} \in O(d)$ with $\hat{q}\eta = \rho$ and $\|\hat{q} - \mathrm{id}\|_{2\to 2} = \|\eta - \rho\|_2$.*

Proof. Let η_2 be a vector in $\mathrm{span}(\eta, \rho)$ which is orthogonal to ρ, and $(\eta_3, \dots, \eta_d)$ be a completion of (η, η_2) to an orthonormal basis. Let further $\rho = \cos\alpha\, \eta + \sin\alpha\, \eta_2$. We then define $\hat{q}$ through the following matrix representation in the basis $(\eta, \eta_2, \dots, \eta_d)$:

$$M = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & \mathrm{id} \end{bmatrix}.$$

This obviously yields $q\eta = \rho$.

To calculate $\|\hat{q} - \mathrm{id}\|_{2\to 2}$, we use the fact that since the basis $(\eta, \eta_2, \dots, \eta_d)$ is orthonormal, $\|q - \mathrm{id}\|_{2\to 2} = \|M - \mathrm{id}\|_{2\to 2}$. If $v \in \mathbb{R}^d$, we have

$$\begin{aligned} \|(M - \mathrm{id})v\|_2^2 &= ((\cos\alpha - 1)v_1 - \sin\alpha v_2)^2 + ((\cos\alpha - 1)v_1 + \sin\alpha v_2)^2 \\ &= ((\cos\alpha - 1)^2 + \sin^2\alpha)(v_1^2 + v_2^2) = 2(1 - \cos\alpha)(v_1^2 + v_2^2), \end{aligned}$$

and hence, $\|M - \mathrm{id}\|_{2\to 2} = \sqrt{2(1 - \cos\alpha)}$. Since at the same time $\|\rho - \eta\|_2^2 = 1 - 2\langle \rho, \eta\rangle + 1 = 2(1 - \cos\alpha)$, the claim has been proven. □

Now we are finally in the situation to prove Lemma 3.2.

Proof of Lemma 3.2. As we already mentioned, it is sufficient to prove that the concentration function $\alpha_{G(d,k)}$ satisfies $\alpha_{G(d,k)}(r) \leq C\exp(-adr^2)$ for some C and $a > 0$. To do this, fix $L \in G(d, k)$ and $\nu \in \mathbb{S}^{d-1}$. Due to (10), we then have, for each Borel set $A \subseteq G(d, k)$,

$$\mu(A) = \sigma^{d-1}\left(\{q\nu \mid q \in O(d) : qL \in A\}\right) \quad \text{and} \quad \mu(A_\epsilon) = \sigma^{d-1}\left(\{q\nu \mid q \in O(d) : qL \in A_\epsilon\}\right).$$

Next, let $\mu(A) \geq 1/2$ and denote $S := \{q\nu \mid q \in O(d) : qL \in A\}$ as well as $S^\epsilon := \{q\nu \mid q \in O(d) : qL \in A_\epsilon\}$. We now claim that

$$\left\{\eta \;\middle|\; d_{\mathbb{S}}(\eta, S) \leq \frac{1}{2}\epsilon\right\} \subseteq S^\epsilon. \tag{11}$$

This will finish the proof, since

$$\mu(A^\epsilon) = \sigma^{d-1}(S^\epsilon) \geq \theta\left(\left\{\eta \;\middle|\; d_{\mathbb{S}}(\eta, S) \leq \frac{1}{2}\epsilon\right\}\right) \geq 1 - \alpha_{\mathbb{S}^{d-1}}\left(\frac{r}{2}\right) \leq C\exp\left(-\frac{ar^2}{4}\right),$$

where we used 9 together with the fact that $\sigma^{d-1}(S) = \mu(A) \geq 1/2$.

Thus it remains to prove (11). For this, let $\rho \in S^\epsilon$. Then there exists some $\eta \in S$ with $\|\rho - \eta\|_2 \leq \epsilon/2$. The vector η can, by definition of S, be represented as $q\nu$, where $qL \in A$. Lemma 5.5 further implies the existence of some $\hat{q} \in O(d)$ with $\rho = \hat{q}\eta = \hat{q}q\nu$ and $\|\hat{q} - \mathrm{id}\|_{2\to 2} \leq \epsilon/2$. Hence

$$\begin{aligned} \mathrm{dist}(\hat{q}qL, A) &\leq d(\hat{q}qL, qL) = \|\Pi_{\hat{q}qL} - \Pi_{qL}\|_{2\to 2} = \|\hat{q}\Pi_{qL}\hat{q}^* - \Pi_{qL}\|_{2\to 2} \\ &\leq \|\hat{q}\Pi_{qL}\hat{q}^* - \Pi_{qL}\hat{q}^*\|_{2\to 2} + \|\Pi_{qL}\hat{q}^* - \Pi_{qL}\|_{2\to 2} \leq 2\,\|\hat{q} - \mathrm{id}\|_{2\to 2} \leq \epsilon. \end{aligned}$$

This finally implies that $\hat{q}qL \in A_\epsilon$, i.e., $\rho \in S^\epsilon$, and the claim has been proven. □

5.3 Proof of Theorem 3.4

Before presenting the rigorous proof of the "global" performance of the index selection procedure of $PROMP$, which inevitably will be a bit cluttered, let us briefly first present its key ideas. We first recall Lemma 3.1, which states that

$$\Pi_{\ker A^\perp} x_0 = R^2 x_0 + R\sqrt{1-R^2} Q_{x_0}[\theta, 0].$$

In order to prove that the procedure will not choose as many as n_1 indices in S_0^c, or will fail to choose n_2 indices in S_0, it suffices to prove that, for each $T_1 \subseteq S_0^c$ with $|T_1| = n_1$ and $T_2 \subseteq S_0$ with $|T_2| = n_2$, we have

$$\left\| \frac{d}{m} \Pi_{\mathbb{R}^{T_1}} \Pi_{\ker A^\perp} x_0 \right\|_{-\infty} \leq \vartheta \quad \text{and} \quad \left\| \frac{d}{m} \Pi_{\mathbb{R}^{T_2}} \Pi_{\ker A^\perp} x_0 \right\|_{\infty} \geq \vartheta.$$

We first use $d/m \cdot R^2 \approx 1$, $\Pi_{R^{T_1}} x_0 = 0$, and $T_2 \subseteq \operatorname{supp} x_0$, to conclude that with very high probability

$$\left\| \frac{d}{m} \Pi_{\mathbb{R}^{T_1}} \Pi_{\ker A^\perp} x_0 \right\|_{-\infty} \approx \left\| \Pi_{\mathbb{R}^{T_1}} \sqrt{d/m - 1} Q_{x_0}[\theta, 0] \right\|_{-\infty} \quad \text{and}$$
$$\left\| \frac{d}{m} \Pi_{\mathbb{R}^{T_2}} \Pi_{\ker A^\perp} x_0 \right\|_{\infty} \approx \left\| \Pi_{\mathbb{R}^{T_2}} x_0 - \Pi_{\mathbb{R}^{T_2}} \sqrt{d/m - 1} Q_{x_0}[\theta, 0] \right\|_{\infty} \geq \|x_0\|_{-\infty} - \left\| \Pi_{\mathbb{R}^{T_2}} \sqrt{d/m - 1} Q_{x_0}[\theta, 0] \right\|_{\infty}.$$

Then we argue, using different types of concentration of measure arguments, that with very high probability

$$\|\Pi_{\mathbb{R}^{T_1}} Q_{x_0}[\theta, 0]\|_{-\infty} \leq \frac{1}{\sqrt{d}} \quad \text{and} \quad \|\Pi_{\mathbb{R}^{T_2}} Q_{x_0}[\theta, 0]\|_{\infty} \approx \sqrt{\frac{d}{d-1}} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)} \left(\sqrt{\log n_2} + \sqrt{\frac{2}{\pi}} \frac{\|x_0\|_\infty}{\|x_0\|_2} \right),$$

which, together with a union bound over all possible choices for T_1 and T_2, principally yields the statement of the theorem.

We will need the following well known result. A proof can be found in, for instance [26], (note that the lemma is formulated slightly different there, but the proof shows that the formulation below is equivalent).

Lemma 5.6. *[26, Lemma 2.6] Let θ be uniformly distributed over the sphere $\mathbb{S}^{d-1}$, and L a fixed m-dimensional subspace. Then there exist universal constants M and α such that*

$$\mathbb{P}\left(\left| \|\Pi_L \theta\|_2 - \sqrt{\frac{m}{d}} \right| \geq t\sqrt{\frac{m}{d}} \right) \leq M \exp(-m\alpha t^2).$$

We now refine our sketch, starting with a lemma concerning estimates of $\|\Pi_{\mathbb{R}^{T_1}} Q_{x_0}[\theta, 0]\|_{-\infty}$ and $\|\Pi_{\mathbb{R}^{T_2}} Q_{x_0}[\theta, 0]\|_{\infty}$.

Lemma 5.7. *Let $\theta \sim \mathcal{U}\left(\mathbb{S}^{d-2}\right)$ and $x_0 \in \mathbb{R}^d$ be supported on S_0. Then there exist universal constants D and b such that, for every $T_1 \subseteq S_0^c$, $T_2 \subseteq S_0$ and $\tau > 0$,*

$$\|\Pi_{\mathbb{R}^{T_1}} Q_{x_0}[\theta, 0]\|_{-\infty} \leq \sqrt{\frac{1}{d}} \left(1 + \tau\sqrt{\frac{m}{d}} \right)$$

with a probability larger that $1 - D\exp(-mb\tau^2)$, and

$$\|\Pi_{\mathbb{R}^{T_2}} Q_{x_0}[\theta, 0]\|_{\infty} \leq \sqrt{\frac{d}{d-1}} \frac{1}{1 - \sqrt{\frac{m}{d-1}}\tau} \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)} \left(\sqrt{\log|T_2|} + \sqrt{\frac{1}{\pi}} \frac{\|x_0\|_\infty}{\|x_0\|_2} \right) + \tau\sqrt{\frac{m}{d}} \right)$$

with a probability larger than $1 - D\exp(-b\tau^2)$.

Proof. To simplify the notation, we use the abbreviation $\eta_0 = x_0/\|x_0\|_2$. Let us start with the statement about $\Pi_{\mathbb{R}^{T_1}} Q_{x_0}[\theta, 0]$. For this, we first argue that we may choose Q_{x_0} in such a way so that $\Pi_{\mathbb{R}^{T_1}} Q_{x_0}[\theta, 0] = [\Pi_{\mathbb{R}^{T_1}}\theta, 0]$. To see this, note that as long as $Q_{x_0} e_d = \eta_0$, the choice is arbitrary. If, without loss of generality, $S_0 = [d - |S_0| + 1, d]$, we may hence ensure that it has this form:

$$Q_{x_0} = \begin{bmatrix} \mathrm{id} & 0 \\ 0 & \tilde{Q}_{x_0} \end{bmatrix}.$$

For this choice of Q_{x_0}, it is evident that $\Pi_{\mathbb{R}^{T_1}} Q_{x_0}[\theta, 0] = [\Pi_{\mathbb{R}^{T_1}}\theta, 0]$.

Next, we bound the probability that $\|\Pi_{\mathbb{R}^{T_1}} Q_{x_0}[\theta, 0]\|_{-\infty} = \|\Pi_{\mathbb{R}^{T_1}}\theta\|_{-\infty} \geq t$. Notice that $\|\Pi_{\mathbb{R}^{T_1}}\theta\|_{-\infty} \geq t$ implies that $\|\Pi_{\mathbb{R}^{T_1}}\theta\|_2 \geq \sqrt{|T_1|}t$. If $t = \sqrt{1/d}(1 + \tau\sqrt{m/|T_1|})$, by Lemma 5.6, this in turn happens with a probability smaller than $M\exp(-m\alpha\tau^2)$. The first claim is proved.

The second claim is a bit trickier to prove. Let us begin by showing that we can write the random variable $Q_{x_0}[\theta, 0]$ a bit differently: If $\nu \sim \mathcal{U}\left(\mathbb{S}^{d-1}\right)$, we have

$$Q_{x_0}\left(\begin{bmatrix}\theta \\ 0\end{bmatrix}\right) \sim \frac{\Pi_{x_0^\perp}\nu}{\|\Pi_{x_0^\perp}\nu\|_2}, \tag{12}$$

where $x_0^\perp$ denotes the orthogonal complement of $\operatorname{span} x_0$. To prove this claim, it is sufficient to prove that $Q_{x_0}^*\Pi_{x_0^\perp}\nu/\|\Pi_{x_0^\perp}\nu\|_2$ is uniformly distributed over $\mathbb{S}^{d-2}$.

For this, first we observe that the variable is almost always well-defined, since $\|\Pi_{x_0^\perp}\nu\|_2 > 0$ with probability 1. Trivially, it has norm 1, and furthermore

$$\left\langle Q_{x_0}^*\Pi_{x_0^\perp}\nu, e_d\right\rangle = \left\langle \Pi_{x_0^\perp}\nu, Q_{x_0}e_d\right\rangle = \left\langle \Pi_{x_0^\perp}\nu, \eta_0\right\rangle = 0,$$

since η_0 and x_0 are parallel. This means that the d-coordinate of $Q_{x_0}^*\Pi_{x_0^\perp}\nu$ is vanishing, i.e., $Q_{x_0}^*\Pi_{x_0^\perp}\nu \in \mathbb{S}^{d-2}$. To prove the uniform distribution, it is sufficient to show that for each u in the orthogonal group $O(d-1)$, we have $uQ_{x_0}^*\Pi_{x_0^\perp}\nu/\|\Pi_{x_0^\perp}\nu\|_2 \sim Q_{x_0}^*\Pi_{x_0^\perp}\nu/\|\Pi_{x_0^\perp}\nu\|_2$. Let us identify u with the matrix in $u \in O(d)$ which acts on $\{e_d\}^\perp$ as does u on $\mathbb{R}^{d-1}$, and leaves e_d invariant. Next we calculate

$$\begin{aligned} uQ_{x_0}^*\Pi_{x_0^\perp}\nu &= uQ_{x_0}^*(\nu - \langle\nu, \eta_0\rangle\eta_0) = uQ_{x_0}^*\nu - \langle\nu, \eta_0\rangle uQ_{x_0}^*\eta_0 = uQ_{x_0}^*\nu - \left\langle uQ_{x_0}^*\nu, e_d\right\rangle e_d \\ &\sim \nu - \langle\nu, e_d\rangle e_d = Q_{x_0}^*\left(Q_{x_0}\nu - \langle Q_{x_0}\nu, \eta_0\rangle Q_{x_0}e_d\right) \sim Q_{x_0}^*\left(\nu - \langle\nu, \eta_0\rangle\eta_0\right) = Q_{x_0}^*\Pi_{x_0^\perp}\nu. \end{aligned}$$

We remark that in this computation, we exploited several times that $q\nu \sim \nu$ for any orthogonal matrix $q \in O(d)$, and also $\eta_0 = Q_{x_0}e_d$. Summarizing, we have

$$\frac{uQ_{x_0}^*\Pi_{x_0^\perp}\nu}{\|\Pi_{x_0^\perp}\nu\|_2} = \frac{uQ_{x_0}^*\Pi_{x_0^\perp}\nu}{\|uQ_{x_0}^*\Pi_{x_0^\perp}\nu\|_2} \sim \frac{Q_{x_0}^*\Pi_{x_0^\perp}\nu}{\|Q_{x_0}^*\Pi_{x_0^\perp}\nu\|_2} = \frac{Q_{x_0}^*\Pi_{x_0^\perp}\nu}{\|\Pi_{x_0^\perp}\nu\|_2},$$

thereby proving (12).

Now we have

$$\left\|\Pi_{\mathbb{R}^{T_2}} Q_{x_0}\left(\begin{bmatrix}\theta \\ 0\end{bmatrix}\right)\right\|_\infty = \frac{\|\Pi_{\mathbb{R}^{T_2}}\Pi_{x_0^\perp}\nu\|_\infty}{\|\Pi_{x_0^\perp}\nu\|_2}.$$

The denominator of this quotient is the ℓ_2-norm of the projection of a uniformly distributed point on $\mathbb{S}^{d-1}$ onto a subspace of dimension $(d-1)$. Therefore, due to Lemma 5.6, we have

$$\mathbb{P}\left(\|\Pi_{x_0^\perp}\nu\|_2 \geq \sqrt{\frac{d-1}{d}}\left(1 - \sqrt{\frac{m}{d-1}}\tau\right)\right) \geq 1 - M\exp(-\alpha m\tau^2). \tag{13}$$

M and α are as in said lemma.

For the numerator, we first estimate its expected value. Note that, if $\rho \perp\!\!\!\perp \nu$ is χ_2-distributed, $\rho\nu =: g$ is Gaussian. Therefore,

$$\mathbb{E}\left(\|\Pi_{\mathbb{R}^{T_2}}\Pi_{x_0^\perp}\nu\|_\infty\right) = \frac{1}{\mathbb{E}\left(\rho\right)}\mathbb{E}\left(\|\Pi_{\mathbb{R}^{T_2}}\Pi_{x_0^\perp}g\|_\infty\right).$$

It is well known that $\mathbb{E}\left(\rho\right) = \sqrt{2}\Gamma(d+1/2)/\Gamma(d/2)$. We furthermore have

$$\mathbb{E}\left(\|\Pi_{\mathbb{R}^{T_2}}\Pi_{x_0^\perp}g\|_\infty\right) \leq \mathbb{E}\left(\|\Pi_{\mathbb{R}^{T_2}}g\|_\infty\right) + \mathbb{E}\left(|\langle g, \eta_0\rangle|\right)\|\Pi_{\mathbb{R}^{T_2}}\eta_0\|_\infty.$$

For the first term in this sum, we derive $\mathbb{E}\left(\|\Pi_{\mathbb{R}^{T_2}}g\|_\infty\right) \leq \sqrt{2\log|T_2|}$, since $\Pi_{\mathbb{R}^{T_2}}g$ is a $|T_2|$-dimensional Gaussian [23]. For the second, we trivially have $\mathbb{E}\left(|\langle g, \eta_0\rangle|\right) = \sqrt{2/\pi}$. Putting everything together, we obtain

$$\mathbb{E}\left(\|\Pi_{\mathbb{R}^{T_2}}\Pi_{x_0^\perp}g\nu\|_\infty\right) \leq \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}\left(\sqrt{\log|T_2|} + \sqrt{\frac{1}{\pi}\frac{\|x_0\|_\infty}{\|x_0\|_2}}\right).$$

Finally, since the function $\eta \to \mathbb{E}\left(\|\Pi_{\mathbb{R}^{T_2}}\Pi_{x_0^\perp}\eta\|_\infty\right)$ is 1-Lipschitz on the sphere, we may apply Theorem 5.3 to conclude that

$$\begin{aligned}\mathbb{P}&\left(\|\Pi_{\mathbb{R}^{T_2}}\Pi_{x_0^\perp}g\nu\|_\infty > \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}\left(\sqrt{\log|T_2|} + \sqrt{\frac{1}{\pi}\frac{\|x_0\|_\infty}{\|x_0\|_2}}\right) + \tau\sqrt{\frac{m}{d}}\right) \\ &\leq \mathbb{P}\left(\|\Pi_{\mathbb{R}^{T_2}}\Pi_{x_0^\perp}g\nu\|_\infty > \mathbb{E}\left(\|\Pi_{\mathbb{R}^{T_2}}\Pi_{x_0^\perp}g\nu\|_\infty\right) + \tau\sqrt{\frac{m}{d}}\right) \leq C\exp\left(-am\tau^2\right). \end{aligned} \tag{14}$$

Putting (13) and (14) together, and setting $D = \max(M, C)$, $b = \min(\alpha, a)$, yields the claim. ☐

With the preparation at hand, the proof of the theorem is now relatively easy.

Proof of Theorem 3.4. We follow the strategy sketched above and consider for $T_1 \subseteq S_0^c$ with $|T_1| = n_1$ and $T_2 \subseteq S_0$ with $|T_2| = n_2$ the random variables $\left\|\frac{d}{m}\Pi_{\mathbb{R}^{T_1}}\Pi_{\ker A^\perp}x_0\right\|_{-\infty}$ and $\left\|\frac{d}{m}\Pi_{\mathbb{R}^{T_2}}\Pi_{\ker A^\perp}x_0\right\|_\infty$. By Lemma 3.1, i.e.,

$$\Pi_{\ker A^\perp}x_0 = R^2x_0 + R\sqrt{1-R^2}Q_{x_0}[\theta, 0],$$

and using the fact that $\Pi_{\mathbb{R}^{T_1}}x_0 = 0$ and $\left\|\Pi_{\mathbb{R}_2^T}x_0\right\|_\infty \geq \|x_0\|_{-\infty}$, we obtain

$$\begin{aligned}\left\|\frac{d}{m}\Pi_{\mathbb{R}^{T_1}}\Pi_{\ker A^\perp}x_0\right\|_{-\infty} &\leq \frac{d}{m}R\sqrt{1-R^2}\left\|\frac{d}{m}\Pi_{\mathbb{R}^{T_1}}Q_{x_0}[\theta, 0]\right\|_{-\infty}, \\ \left\|\frac{d}{m}\Pi_{\mathbb{R}^{T_2}}\Pi_{\ker A^\perp}x_0\right\|_\infty &\geq \frac{d}{m}R^2\|x_0\|_{-\infty} - R\sqrt{1-R^2}\|\Pi_{\mathbb{R}^{T_2}}Q_{x_0}[\theta, 0]\|_\infty.\end{aligned}$$

This implies that if the events

- $E_R = \left\{\sqrt{\frac{d}{m}R} \in (1-\tau, 1+\tau)\right\}$,
- $E_{T_1} = \left\{\|\Pi_{\mathbb{R}^{T_1}}Q_{x_0}[\theta, 0]\|_{-\infty} \leq \sqrt{\frac{|T_1|}{d}}\left(1+\tau\sqrt{\frac{m}{d}}\right)\right\}$, and
- $E_{T_2} = \left\{\|\Pi_{\mathbb{R}^{T_2}}Q_{x_0}[\theta, 0]\|_\infty \leq \sqrt{\frac{d}{(d-1)\left(1-\sqrt{\frac{m}{d-1}}\tau\right)}}\left(\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}\left(\sqrt{\log|T_2|} + \sqrt{\frac{1}{\pi}\frac{\|x_0\|_\infty}{\|x_0\|_2}}\right) + \tau\sqrt{\frac{m}{d}}\right)\right\}$

occur, we have

$$\left\| \frac{d}{m} \Pi_{\mathbb{R}^{T_1}} \Pi_{\ker A^\perp} x_0 \right\|_{-\infty} \leq \vartheta_-(\tau, n_1, m) \quad \text{and} \quad \left\| \frac{d}{m} \Pi_{\mathbb{R}^{T_2}} \Pi_{\ker A^\perp} x_0 \right\|_{\infty} \geq \vartheta_+(\tau, n_2, m),$$

where we defined

$$\begin{aligned}
\vartheta_-(\tau, n_1, m) &:= (1+\tau) \sqrt{\frac{d}{m} - (1-\tau)^2} \sqrt{\frac{|T_1|}{d}} \left(1 + \tau \sqrt{\frac{m}{d}}\right) \\
\vartheta_+(\tau, n_2, m) &:= (1-\tau)^2 \|x_0\|_\infty - (1+\tau) \sqrt{\frac{d}{m} - (1-\tau)^2} \cdot \sqrt{\frac{d}{d-1}} \frac{1}{1 - \sqrt{\frac{m}{d-1}}\tau} \\
&\quad \left(\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)} \left(\sqrt{\log |T_2|} + \sqrt{\frac{1}{\pi}} \frac{\|x_0\|_\infty}{\|x_0\|_2} \right) + \tau \sqrt{\frac{m}{d}} \right).
\end{aligned}$$

Now, according to the definition of R and Lemma 5.6,

$$\mathbb{P}(E_R) \geq \left(1 - D \exp\left(-am\tau^2\right)\right), \qquad \mathbb{P}\left(E_{T_1}^c\right) \leq D \exp(-b\tau^2/2), \quad \text{and} \quad \mathbb{P}\left(E_{T_2}^c\right) D \exp(-mb\tau^2/2).$$

Since furthermore $E_R \perp\!\!\!\perp (E_{T_1}, E_{T_2})_{T_1 \subseteq S_0^c, T_s \subseteq S_0}$ (since R and θ are independent, we conclude that

$$\begin{aligned}
\mathbb{P}\Bigg(E_R \cap \Big(\bigcap_{\substack{T_1 \subseteq S_0^c \\ |T_1| = n_1}} E_{T_1} \cup \bigcap_{\substack{T_2 \subseteq S_0 \\ |T_2| = n_2}} E_{T_2} \Big) \Bigg) &\geq \mathbb{P}(E_R) \Bigg(1 - \Big(\sum_{\substack{T_1 \subseteq S_0^c \\ |T_1| = n_1}} \mathbb{P}\left(E_{T_1}^c\right) + \sum_{\substack{T_2 \subseteq S_0 \\ |T_2| = n_2}} \mathbb{P}\left(E_{T_2}^c\right) \Big) \Bigg) \\
&\geq \left(1 - \exp\left(-bm\tau^2\right)\right) \left(1 - D \binom{d - |S_0|}{n_1} \binom{|S_0|}{n_2} \exp\left(-bm\tau^2\right) \right),
\end{aligned}$$

which is the assertion of the theorem. □

5.4 Proof of Theorem 3.13

We will now carry out the argument which was sketched in Section 3.2. To do that, we require two results from geometric measure theory.

Lemma 5.8. *(i) [3, p.5] Let $g \in \mathbb{R}^d$ be a Gaussian vector and $\epsilon > 0$. Then we have*

$$\mathbb{P}\left(\|g\|_2^2 \leq d(1-\epsilon) \right) \leq \exp(-\epsilon^2/4).$$

and

$$\mathbb{P}\left(\|g\|_2^2 \geq d/(1+\epsilon) \right) \leq \exp(-\epsilon^2/4).$$

(ii) Let θ be uniformly distributed on the sphere $\mathbb{S}^{d-1}$ and $\nu \in \mathbb{S}^{d-1}$ be arbitrary. Then, for every $t > 0$,

$$\mathbb{P}\left(|\langle \theta, \nu \rangle| > t\right) \leq \sqrt{\frac{\pi}{2}} \exp(-t^2/4).$$

Proof. A proof of (i) can be found in [3].

Let us briefly argue why (ii) is true. Due to symmetry, the set $A = \{\theta \mid \langle \theta, \nu \rangle \leq 0\}$ has measure 1/2 (i.e., 0 is a *median* of the function $\theta \to \langle \theta, \nu \rangle$). It is easy to convince oneself that $\{\langle \theta, \nu \rangle \leq t\} \supseteq A_t$. Therefore we have, due to the definition of the concentration function and the decay bound (9),

$$\mathbb{P}\left(\langle \theta, \nu \rangle > t\right) \leq \sigma^{d-1}(A_t) \leq \sqrt{\frac{\pi}{8}} \exp(-t^2/4).$$

Since $\mathbb{P}\left(\langle \theta, \nu \rangle < t\right)$ can be bounded in a similar manner, the proof is finished. □

Now we can can turn to the proof the theorem.

Proof of Theorem 3.13. Since $\Pi_{\operatorname{ran} A_S^\perp} a_i = 0$ for $i \in S$, the values $\mu_S(A)$ and $\max_{i\in I} \left\|\Pi_{\operatorname{ran} A_S^\perp} a_i\right\|_2^2$ are only dependent on the vectors $(a_i)_{i\neq S}$. This together with Lemma 3.11 and Remark 3.12 implies that we may replace $\Pi_{\operatorname{ran} A_S^\perp} a_j$ with $\tilde{a}_j$ in all our calculations, where $\tilde{a}_j$ are the columns of a Gaussian matrix $\tilde{A} \in \mathbb{R}^{m-|S|,d-|S|} := \mathbb{R}^{\tilde{m},\tilde{d}}$. We must hence control the probability that

$$\max_{i\in I} \|\tilde{a}_i\|_2^2 \geq \mu(\tilde{A})(2\,|S_0\backslash S| - 1). \tag{15}$$

To do this, we consider each side of the inequality (15) on its own. The expression on the left hand side can be controlled with the help Lemma 5.8(i) as follows. According to this lemma, the probability than one vector $\tilde{a}_i \in \mathbb{R}^{\tilde{m}}$ has squared ℓ_2-norm smaller than $\tilde{m}(1-\epsilon)$ is smaller than $\exp(-\tilde{m}\epsilon^2/4)$. Since the vectors $(\tilde{a}_i)_{i\in I}$ are independent, we hence have

$$\mathbb{P}\left(\max_{i\in I} \|\tilde{a}_i\|_2^2 \leq \tilde{m}(1-\epsilon)\right) = \mathbb{P}\left(\forall i \in I \ : \ \|\tilde{a}_i\|_2^2 \leq \tilde{m}(1-\epsilon)\right) = \prod_{i\in I} \mathbb{P}\left(\ \|\tilde{a}_i\|_2^2 \leq \tilde{m}(1-\epsilon)\right) \leq \exp(-\tilde{m}\,|I|\,\epsilon^2/4).$$

Let us for now postpone the choice of ϵ and instead consider the right hand side of (15). For each i and j, using the fact that $\tilde{A}$ is Gaussian, we have

$$\langle \tilde{a}_i, \tilde{a}_j\rangle = \|\tilde{a}_i\|_2 \|\tilde{a}_j\|_2 \langle \theta_i, \theta_j\rangle\,,$$

where $(\theta_i)_{i=1\ldots\tilde{d}}$ is a family of independent vectors uniformly distributed over $\mathbb{S}^{\tilde{m}-1}$. Now, due to Lemma 5.8(i), we have $\|\tilde{a}_i\|_2 \leq \sqrt{\tilde{m}/(1-t)}$ with a probability larger than $1 - \tilde{d}\exp(-\tilde{m}t^2/4)$. Moreover, Lemma 5.8(ii) together with the fact that the θ's are independent implies that $|\langle\theta_i,\theta_j\rangle| \leq t$ with a probability larger than $1 - \sqrt{\frac{\pi}{2}}\exp(-t^2/4) \geq 1 - \sqrt{\frac{\pi}{2}}\exp(-\tilde{m}t^2/4)$. A union bound and the above inequality secure that

$$\mathbb{P}\left(\mu(\tilde{A}) \geq \tilde{m}t/(1-t)\right) \leq 1 - \left(\sqrt{\frac{\pi}{2}}\tilde{d}(\tilde{d}-1) + \tilde{d}\right)\exp(-\tilde{m}t^2/4)$$

Summarizing, we just have proven that condition (15) holds with a complementary probability smaller than $\left(\sqrt{\frac{\pi}{2}}\tilde{d}(\tilde{d}-1)+\tilde{d}\right)\exp(-\tilde{m}t^2/4) + \exp(-\tilde{m}\,|I|\,\epsilon^2/4)$, provided that we choose the parameters ϵ, t in such a way that

$$(1-\epsilon) \geq \frac{t}{1-t}(2\,|S_0\backslash S| - 1). \tag{16}$$

Let us now choose $\epsilon = t/\sqrt{|I|}$. Then it is an elementary algebra exercise to prove that (16) is satisfied, if we select t to satisfy

$$0 \leq t \leq \frac{2\,|S_0\backslash S|\,\sqrt{|I|}+1}{2}\left(1 - \sqrt{1 - \sqrt{|I|}\left(\frac{2}{2\,|S_0\backslash S|\,\sqrt{|I|}+1}\right)^2}\right) \leq \frac{2\sqrt{|I|}}{2\,|S_0\backslash S|\,\sqrt{|I|}+1},$$

where we in the last step used that $\sqrt{1-x} \geq 1-x$ for $0 \leq x \leq 1$. Hence, the probability that (15) is fulfilled is larger than

$$1 - \left(e\sqrt{\frac{\pi}{8}}\tilde{d}(\tilde{d}-1) + \tilde{d}\right)\exp\left(-\frac{\tilde{m}}{4}\cdot\left(\frac{2\sqrt{|I|}}{2\,|S_0\backslash S|\,\sqrt{|I|}+1}\right)^2\right).$$

But this term is larger than $1-\eta$, if $\tilde{m} \geq \left(\frac{2|S_0\backslash S|\sqrt{|I|}+1}{2\sqrt{|I|}}\right)^2 \log\left(\frac{\sqrt{\frac{\pi}{2}}\tilde{d}(\tilde{d}-1)+\tilde{d}}{\eta}\right)$, which is equivalent to the assumption of the theorem. $\square$

5.5 Proof of Theorem 3.15

Theorem 3.15 is a generalization of an analogous statement from the paper [43]. The proof follows to some extent the lines of the one in that paper; however at the same time several steps in the argument require careful adaption.

In the sequel, we will make use of the two fairly standard results on Gaussian matrices.

Lemma 5.9. *(i) [43] Let g be Gaussian and $q_1, \dots q_n$ a sequence of vectors with less than unit norm, which is independent of a_i. Then, for every $t > 0$, we have*

$$\mathbb{P}\left(\max_i |\langle g, q_i\rangle| \geq t\right) \geq (1 - \exp(-t^2/2))^k.$$

(ii) [12] Let $A \in \mathbb{R}^{m,d}$ be Gaussian, and denote its smallest non-zero singular value with $\sigma_{\min}(A)$. Then, for every $\epsilon > 0$, we have

$$\mathbb{P}\left(\sigma_{\min}(A) > \sqrt{d} - \sqrt{m} - \epsilon\right) \leq \exp(-\epsilon^2/2)$$

Remark 5.10. *We wish to remark that part (ii) of this lemma is often stated for matrices $\sim \mathcal{N}(0, d^{-1}\,\mathrm{id})$. One may although of course switch to the standard Gaussian setting simply by scaling.*

With the help of this lemma, we can now present the proof of Theorem 3.15.

Proof of Theorem 3.15. Let us begin by noting that in order to prove that x_0 is recovered by OMP, it suffices to show that the indices in $S_0\backslash S$ are found by OMP in $|S_0\backslash S|$ iterations. If this happens, the $|S_0\backslash S|$-th support iterate S_* will have not more than m indices (due to the assumption $|S| + |S\backslash S_0| \leq m$). Therefore, the Gaussian matrix A_{S_*} formed by the columns corresponding to S_* will almost surely be injective, and hence, the equation $Ax = b$ will have at most one solution on $\mathbb{R}^{S_*}$. Since $S_0 \subseteq S_*$ due to assumption, the solution x_0 of $Ax = b$ will lie in $\mathbb{R}^{S_*}$, and will hence be the only solution. Thus, x_0 will be uniquely reconstructed in $|S_0 \neq S|$ iterations.

After this preconsideration, we now start the proof by decomposing the matrix A into three parts:

$$A = [A_S \mid A_{S_0\backslash S} \mid A_{S_0 \cup S^c}].$$

The three parts of the matrix are independent and all are Gaussian. To enhance readability of the rest of the argument, let us abbreviate

$$T := S_0\backslash S \quad \text{and} P := (S_0 \cup S)^c.$$

Also, define the *selection ratio*

$$\rho : \mathbb{R}^m \to \mathbb{R}, r \mapsto \frac{\|A_P^* r\|_\infty}{\|A_T^* r\|_\infty}.$$

According to the selection procedure of OMP, an index in T (i.e., a correct index) will be chosen if and only if the current residual r obeys $\rho(r) < 1$.

Now we use the very clever technique of a "virtual residual sequence" (q_k) from [43]. We define this sequence as the sequence of residuals which would appear, if we ran OMP with the matrix $[A_S \mid A_T]$. Due to the fact that, if OMP is successful at recovering the indices in T, it will choose exactly that sequence, and this will happen if and only if $\rho(q_k) < 1$ for each k, we conclude

$$\mathbb{P}(\ OMP \text{ finds all indices in } T) = \mathbb{P}\left(\max_{k=1\dots|T|} \rho(q_k) < 1\right).$$

(For a more detailed argument, we refer to [43].) By the definition of the q_k's, they will all be contained in the column space of $[A_S \mid A_T]$ and also be orthogonal to $\operatorname{ran} A_S$, i.e., $q_k \in \operatorname{ran} \Pi_{\operatorname{ran} A_S^\perp} A_T$.

To estimate the probability of the event $E = \left\{\max_{k=1\ldots|T|} \rho(q_k) < 1\right\}$, we use that, for every $\sigma > 0$,

$$\mathbb{P}(E) \geq \mathbb{P}\left(E \cap \left\{\sigma_{\min}(\Pi_{\operatorname{ran} A_S^\perp} A_T) \geq \sigma\right\}\right) = \mathbb{P}\left(E \mid \Sigma\right)\mathbb{P}\left(\Sigma\right), \tag{17}$$

where we abbreviated $\Sigma := \left\{\sigma_{\min}(\Pi_{\operatorname{ran} A_S^\perp} A_T) \geq \sigma\right\}$. Next we bound the probability of each of the factors on the right hand side of (17), starting with $\mathbb{P}(\Sigma)$. According to Lemma 3.11 and Remark 3.12, we have

$$\mathbb{P}\left(\Sigma\right) = \mathbb{P}\left(\sigma_{\min}(\tilde{A}) \geq \sigma\right),$$

where $\tilde{A} \in \mathbb{R}^{\tilde{m},|T|}$, $\tilde{m} = m - |S|$ is Gaussian. The probability of this event can furthermore be bounded below with the help of Lemma 5.9(ii). In fact, choosing $\sigma = \sqrt{\tilde{m}} - \sqrt{|T|} - \epsilon$, we obtain

$$\mathbb{P}\left(\Sigma\right) \geq 1 - \exp(-\epsilon^2/2). \tag{18}$$

Let us now turn to the other factor from (17). Since $A_T^* q_k \in \mathbb{R}^{|T|}$, we have $\|A_T^* q_k\|_2 \leq \sqrt{|T|}\,\|A_T^* q_k\|_\infty$. Furthermore, since $q_k \in \operatorname{ran}(\Pi_{\operatorname{ran} A_S^\perp} A_T)$, we have $\|A_T^* q_k\|_2 = \left\|A_T^* \Pi_{\operatorname{ran} A_S^\perp} q_k\right\|_2 \geq \sigma_{\min}(\Pi_{\operatorname{ran} A_S^\perp} A_T)\,\|q_k\|_2 \geq \sigma\,\|q_k\|_2$. Hence,

$$\rho(q_k) = \frac{\|A_P^* q_k\|_\infty}{\|A_T^* q_k\|_\infty} \leq \frac{\sqrt{|T|}}{\sigma} \cdot \frac{\max_{\ell \in P} |\langle a_\ell, q_k\rangle|}{\|q_k\|_2}.$$

Thus, setting $u_k := q_k / \|q_k\|_2$, we conclude that these vectors are all normalized, orthogonal to $\operatorname{ran} A_S$ and independent of $(a_i)_{i \in P}$. In particular,

$$\max_{k \in T} \rho(q_k) \leq \max_k \frac{\sqrt{|T|}}{\sigma} \cdot \max_{\ell \in P} \left|\left\langle \Pi^\perp_{\operatorname{ran} A_S} a_\ell, u_k\right\rangle\right|.$$

Next we would like to apply Lemma 3.11. This is although not possible, since the function $A \mapsto \max_{\ell \in P} \left|\left\langle \Pi_{\operatorname{ran} A_S^\perp} a_\ell, u_k\right\rangle\right|$ is not invariant under orthogonal transformations, even if we condition on the u_k. We can however still use the philosophy of the result as follows: Since $(a_i)_{i \in P} \perp\!\!\!\perp (a_i)_{i \in S \cup T}$, we may condition on $(a_i)_{i \in S \cup T}$, i.e., see them as fixed. Let $q \in O(m)$ be such that $\operatorname{ran} A_S^\perp = qL_n = q \operatorname{span}(e_1, \ldots e_n)$. Then

$$\max_{k \in T} \max_{\ell \in P} \left|\left\langle \Pi^\perp_{\operatorname{ran} A_S} a_\ell, u_k\right\rangle\right| = \max_{k \in T} \max_{\ell \in P} \left|\langle \Pi_{L_n} q^* a_\ell, q^* u_k\rangle\right| \sim \max_{k \in T} \max_{\ell \in P} \left|\langle \tilde{a}_\ell, \tilde{u}_k\rangle\right|,$$

where $(\tilde{a}_\ell)_{\ell \in P}$ is the columns of a Gaussian matrix $\tilde{A} \in \mathbb{R}^{\tilde{m},|P|}$ (see also the proof of Lemma 3.11) and $\tilde{u}_k = q^* u_k$ is a sequence of normalized vectors independent of $(\tilde{a}_\ell)_{\ell \in P}$. Hence, by interchanging the maxima and exploiting the independence of the $(\tilde{a}_i)_i$, we obtain

$$\mathbb{P}\left(E \mid \sigma\right) = \mathbb{P}\left(\max_{k \in T} \max_{\ell \in P} |\langle \tilde{a}_\ell, \tilde{u}_k\rangle| < \frac{\sigma}{\sqrt{T}} \mid \Sigma\right) = \prod_{\ell \in P} \mathbb{P}\left(\max_{k \in T} |\langle \tilde{a}_\ell, \tilde{u}_k\rangle| < \frac{\sigma}{\sqrt{T}} \mid \Sigma\right) \tag{19}$$

$$\geq \left(1 - \exp\left(-\frac{\sigma^2}{2\,|T|}\right)\right)^{|T||P|}, \tag{20}$$

where in the last step we used Lemma 5.9(i).

Concluding, by applying (18) and (19) to (17) and remembering that we chose $\sigma = \sqrt{\tilde{m}} - \sqrt{|T|} - \epsilon$, we obtain

$$\mathbb{P}\left(E\right) \geq \left(1 - \exp\left(-\frac{(\sqrt{\tilde{m}} - \sqrt{|T|} - \epsilon)^2}{2\,|T|}\right)\right)^{|T||P|} (1 - \exp(-\epsilon^2/2)). \tag{21}$$

It now remains to choose ϵ wisely. Noticing that $|T| + |P| = |S_0 \backslash S| + d - |S \cup S_0| = d - |S|$, we may use exactly the same argument as the authors of [43] to deduce that it is possible to choose ϵ in such a manner so that the right hand side of (21) is larger than $1 - \eta$ provided that

$$\tilde{m} \geq C\,|T| \log(\tilde{d}/\eta)$$

with $C = 4(1 + \tilde{m}^{-1/2})$. This is exactly the statement of the theorem. □

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