AFFINE DENSITY, FRAME BOUNDS, AND THE ADMISSIBILITY CONDITION FOR WAVELET FRAMES

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Abstract. For a large class of irregular wavelet frames we derive a fundamental relationship between the affine density of the set of indices, the frame bounds, and the admissibility constant of the wavelet. Several implications of this theorem are studied. For instance, this result reveals one reason why wavelet systems do not display a Nyquist phenomenon analogous to Gabor systems, a question asked in Daubechies' *Ten Lectures* book. It also implies that the affine density of the set of indices associated with a tight wavelet frame has to be uniform. Finally, we show that affine density conditions can even be used to characterize existence of wavelet frames, thus serving in particular as sufficient conditions.

1. Introduction

Provided that a wavelet $\psi \in L^2(\mathbb{R})$ gives rise to a classical wavelet frame $\{a^{-j/2}\psi(a^{-j}x - bk) : j,k \in \mathbb{Z}\}$ with parameters $a > 1$, $b > 0$ and with frame bounds A, B, a result by Chui and Shi [4] and by Daubechies [7] establishes the following intriguing relationship between the parameters, the frame bounds, and the admissibility constant $\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\, d\omega$:

$$A \le \frac{1}{2b\ln a}\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\, d\omega \le B. \tag{1}$$

In particular, this leads to the exact value of the frame bound for tight classical wavelet frames in terms of the parameters and the admissibility constant.

In this paper we discuss necessary and sufficient conditions for wavelets to generate a frame involving the explicit values of the frame bounds in a much more general setting motivated by the following observation. Provided that $\{a^{-j/2}\psi(a^{-j}x - bk) : j,k \in \mathbb{Z}\}$ forms a frame for $L^2(\mathbb{R})$, then each $f \in L^2(\mathbb{R})$ can be reconstructed in a numerically stable way from the sampling points $\{(a^j, bk) : j,k \in \mathbb{Z}\}$ of the continuous wavelet transform $W_\psi f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$, $W_\psi f(x,y) = \langle f, \frac{1}{\sqrt{x}}\psi(\frac{\cdot}{x} - y)\rangle$. However, sampling points may vary in practice and the question arises whether and how perturbing the set of dilation indices and the set of translation indices affects the results obtained for classical wavelet frames. In particular, we are interested in studying necessary conditions involving the frame bounds and also sufficient conditions depending only on the set of perturbed dilations.

Throughout this paper we will study systems consisting of arbitrary time shifts and arbitrary scale shifts of some wavelet $\psi \in L^2(\mathbb{R})$. In detail, given an arbitrary countable set

Date: November 30, 2005.

1991 *Mathematics Subject Classification.* Primary 42C40; Secondary 42C15, 94A12.

Key words and phrases. Admissibility condition, Affine density, Frame, Frame bounds, Nyquist phenomenon, Wavelet system.

of dilations $S \subset \mathbb{R}^+$ and an arbitrary countable set of translations $T \subset \mathbb{R}$, we will consider irregular wavelet systems with respect to scale-time shifts $\Lambda = S \times T$, regarded as a subset of the affine group $\mathbb{A}$, of the form

$$\mathcal{W}(\psi, \Lambda) = \{\tfrac{1}{\sqrt{s}}\psi(\tfrac{x}{s} - t) : (s,t) \in \Lambda\}. \tag{2}$$

In this context, affine density conditions have turned out to be an especially useful and elegant approach [10, 12, 15, 16]. For wavelet frames of the form (2) we derive, under some mild condition on the set T, the following relation of affine density to the frame bounds:

$$A \le \frac{1}{2}\mathcal{D}^-(\Lambda) \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\, d\omega \le \frac{1}{2}\mathcal{D}^+(\Lambda) \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\, d\omega \le B, \tag{3}$$

where $\mathcal{D}^-(\Lambda)$ and $\mathcal{D}^+(\Lambda)$ denote, respectively, the lower and upper affine density as defined in [10]. This result can indeed be shown to generalize (1). It further generalizes several results known for classical wavelet frames and also leads to new surprising results, for example, showing that the affine density of a tight wavelet frame (2) necessarily has to be uniform, i.e., $\mathcal{D}^-(\Lambda) = \mathcal{D}^+(\Lambda)$. Recently, some results in the same spirit have been derived for Gabor frames [2, 13].

Interestingly, this result has direct impact on the well-known question initially stated by Daubechies [7, Sec. 4.1], namely, why wavelet systems do not satisfy a Nyquist phenomenon analogous to Gabor systems. In short, in terms of necessary conditions for Gabor frames there is a critical or Nyquist density for the set of indices separating frames from non-frames, and furthermore the Riesz bases sit exactly at this critical density (compare [3, 14]). It is natural to ask whether wavelet systems share similar properties, and the immediate answer is that there is clearly no exact analogue of the Nyquist density for wavelets, even given constraints on the norm or on the admissibility condition of the wavelet, see the example of Daubechies in [6, Thm. 2.10] and the more extensive analysis of Balan in [1]. Our result now reveals one reason why there does not exist a critical density for orthonormal wavelet bases. In brief, the set of indices of an orthonormal wavelet basis has indeed uniform affine density. However, (3) implies that for these systems

$$\mathcal{D}^-(\Lambda) = \mathcal{D}^+(\Lambda) = 2\|\hat{\psi}\|^{-2}_{L^2(\mathbb{R}, \frac{d\omega}{|\omega|})},$$

which can be shown to attain each positive value.

Conceptually, density conditions seem to be capable only of delivering necessary conditions for the existence of wavelet frames, since they are independent of the wavelet itself and do not capture local features of the set of indices. Hence they appear almost too weak to serve as a sufficient condition. And in fact, to date, the notion of affine density was only employed to derive necessary conditions. However, we show that under some mild decay condition on a band-limited wavelet ψ, the existence of a set $T \subset \mathbb{R}$ such that $\mathcal{W}(\psi, S \times T)$ constitutes a frame for $L^2(\mathbb{R})$ is in fact equivalent to the set $S \subset \mathbb{R}^+$ having positive lower and finite upper affine density.

This paper is organized as follows. In Section 2 we introduce the notion of density for subsets of $\mathbb{A}$, $\mathbb{R}^+$, and $\mathbb{R}$ and recall the basic notation for frames. Section 3 contains the fundamental theorem (Theorem 3.1) on the relation among the affine density, frame bounds, and the admissibility constant for wavelet frames. Further we discuss several of its applications. A detailed analysis of the impact of this result on the non-existence of the Nyquist phenomenon for wavelet systems is then presented in Remark 3.5. Finally, in Section 4 we study the extent to which density conditions can serve as sufficient conditions for the existence of wavelet frames thereby presenting in Theorem 4.2 a situation, where this indeed can be achieved.

2. Notion of density and frames

2.1. **Density of subsets of the affine group $\mathbb{A}$.** In $\mathbb{R}^n$, Beurling density is a measure of the "average" number of points of a set that lie inside a unit cube. In [10] the authors defined a Beurling density that is suited to the geometry of the affine group with the interesting property that classical wavelet systems $\mathcal{W}(\psi, \{(a^j, bk) : j,k \in \mathbb{Z}\})$, $a > 1$, $b > 0$ enjoy a uniform density equal to the ubiquitous constant $\frac{1}{b \ln a}$ ([10, Prop. 4.3]) as was already conjectured by Daubechies in [7, Sec. 4.1]. Also Sun and Zhou [15] simultaneously introduced a density notion for the affine group, but unlike the one from [10] a weighted form has to be used to derive the same uniform density for the classical wavelet systems. Thus in this paper we will employ the density notion from [10]. For a comparison between the two different notions in [10] and [15] we refer the reader to [12].

Let $\mathbb{A} = \mathbb{R}^+ \times \mathbb{R}$ denote the affine group, endowed with the multiplication $(x,y)\cdot(a,b) = (xa, \frac{y}{a} + b)$. For $h > 0$, we let Q_h denote a fixed family of neighborhoods of the identity element $e = (1,0)$ in $\mathbb{A}$ chosen as $Q_h = [e^{-h}, e^h) \times [-h, h)$. For $(x,y) \in \mathbb{A}$, we define $Q_h(x,y)$ by

$$Q_h(x,y) = (x,y)\cdot Q_h = \{(xa, \tfrac{y}{a} + b) : a \in [e^{-h}, e^h),\ b \in [-h,h)\},$$

and let $\mu = \frac{dx}{x}\,dy$ denote the left-invariant Haar measure on $\mathbb{A}$. Since μ is left-invariant,

$$\mu(Q_h(x,y)) = \mu(Q_h) = \int_{-h}^{h} \int_{e^{-h}}^{e^h} \frac{dx}{x}\,dy = 4h^2.$$

Let Λ be a subset of $\mathbb{A}$. Then the *upper affine density* of Λ is

$$\mathcal{D}^+(\Lambda) = \limsup_{h\to\infty} \frac{\sup_{(x,y)\in\mathbb{A}} \#(\Lambda \cap Q_h(x,y))}{4h^2},$$

and the *lower affine density* of Λ is

$$\mathcal{D}^-(\Lambda) = \liminf_{h\to\infty} \frac{\inf_{(x,y)\in\mathbb{A}} \#(\Lambda \cap Q_h(x,y))}{4h^2}.$$

If $\mathcal{D}^-(\Lambda) = \mathcal{D}^+(\Lambda)$, then we say that Λ has *uniform affine density* and denote this density by $\mathcal{D}(\Lambda)$.

Since in the sequel we will study subsets of $\mathbb{A}$ of the form $\Lambda = S \times T$, where $S \subset \mathbb{R}^+$ and $T \subset \mathbb{R}$, the definition of density for subsets of $\mathbb{R}^+$ and $\mathbb{R}$ will become important. Notice that we use the same notion for all three densities. The type of density is then always completely determined by the set to which it is applied.

2.2. **Density of subsets of $\mathbb{R}^+$.** Adapting the definition of Beurling density to the geometry of the multiplicative group $\mathbb{R}^+$ with Haar measure defined by $\mu = \frac{dx}{x}$ in a similar way as it was done in the previous subsection for the affine group yields the following density notion. For $S \subset \mathbb{R}^+$, the *upper density* of S is

$$\mathcal{D}^+(S) = \limsup_{h \to \infty} \frac{\sup_{x \in \mathbb{R}^+} \#(S \cap x[e^{-h}, e^h))}{2h},$$

and the *lower density* of S is

$$\mathcal{D}^-(S) = \liminf_{h \to \infty} \frac{\inf_{x \in \mathbb{R}^+} \#(S \cap x[e^{-h}, e^h))}{2h}.$$

If $\mathcal{D}^-(S) = \mathcal{D}^+(S)$, then S has *uniform density*, which is denoted by $\mathcal{D}(S)$.

In a similar way as for the affine density [10, Prop. 2.2 and Prop. 2.3], the following two results are useful reinterpretations of finite upper and positive lower density. We include a short proof for the first lemma for completeness. Lemma 2.2 can be proven in a similar way.

Lemma 2.1. *Let $S \subset \mathbb{R}^+$. Then the following conditions are equivalent.*

(i) $\mathcal{D}^+(S) < \infty$.
(ii) *There exists an interval $I \subset \mathbb{R}^+$ with $0 < \mu(I) < \infty$ and some $N_I < \infty$ such that $\#(S \cap xI) < N_I$ for all $x \in \mathbb{R}^+$.*
(iii) *For every interval $I \subset \mathbb{R}^+$ with $0 < \mu(I) < \infty$, there exists $N_I < \infty$ such that $\#(S \cap xI) < N_I$ for all $x \in \mathbb{R}^+$.*

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (ii) are trivial.

In the following we prove (ii) $\Rightarrow$ (i), (iii). Suppose there exists an interval $I \subset \mathbb{R}^+$ with $0 < \mu(I) < \infty$ and some constant $N_I < \infty$ such that $\#(S \cap xI) < N_I$ for all $x \in \mathbb{R}^+$. Let J be another interval with $0 < \mu(J) < \infty$ in $\mathbb{R}^+$. If there exists $y \in \mathbb{R}^+$ with $yJ \subset I$, then $\#(S \cap xJ) < N_I$ for all $x \in \mathbb{R}^+$. On the other hand, if there exists $y \in \mathbb{R}^+$ with $yI \subset J$, then $\mu(J) = r\mu(I)$ for some $r \geq 1$, and J is covered by a union of at most $r + 1$ sets of the form xI. Consequently,

$$\sup_{x \in \mathbb{R}^+} \#(S \cap xJ) \leq (r+1) \sup_{x \in \mathbb{R}^+} \#(S \cap xI) \leq (r+1)N_I.$$

Thus statement (iii) holds. Furthermore, choose $z \in \mathbb{R}^+$ and $h > 1$ with $z[e^{-h}, e^h) = I$. Then there exists $a \in \mathbb{R}^+$ so that $J = az[e^{-rh}, e^{rh})$. Hence

$$\mathcal{D}^+(S) \leq \limsup_{r \to \infty} \frac{\sup_{x \in \mathbb{R}^+} \#(S \cap xJ)}{2rh} \leq \limsup_{r \to \infty} \frac{(r+1)N_I}{2rh} = \frac{N_I}{2h} < \infty,$$

so statement (i) holds as well. □

Lemma 2.2. *Let $S \subset \mathbb{R}^+$. Then the following conditions are equivalent.*

(i) $\mathcal{D}^-(S) > 0$.
(ii) *There exist an interval $I \subset \mathbb{R}^+$ with $0 < \mu(I) < \infty$ and some $N_I > 0$ such that $\#(S \cap xI) > N_I$ for all $x \in \mathbb{R}^+$.*

The following result shows that this density is robust against perturbations.

Lemma 2.3. *Let $S \subset \mathbb{R}^+$ and $\epsilon > 0$. For each $\tilde{S} = \{\delta_s s : s \in S,\, \delta_s \in [e^{-\epsilon}, e^{\epsilon}]\}$, we have $\mathcal{D}^-(S) = \mathcal{D}^-(\tilde{S})$ and $\mathcal{D}^+(S) = \mathcal{D}^+(\tilde{S})$.*

Proof. For $h > 0$ and $x \in \mathbb{R}^+$, we obtain the following estimates for $\#(S \cap x[e^{-h}, e^h))$:

$$\#(\tilde{S} \cap x[e^{-(h-\epsilon)}, e^{h-\epsilon})) \le \#(S \cap x[e^{-h}, e^h)) \le \#(\tilde{S} \cap x[e^{-(h+\epsilon)}, e^{h+\epsilon})).$$

Dividing the terms by $2h$, and observing that

$$\limsup_{h\to\infty} \frac{\sup_{x\in\mathbb{R}^+} \#(\tilde{S} \cap x[e^{-(h-\epsilon)}, e^{h-\epsilon}))}{2h} = \mathcal{D}^+(\tilde{S}) = \limsup_{h\to\infty} \frac{\sup_{x\in\mathbb{R}^+} \#(\tilde{S} \cap x[e^{-(h+\epsilon)}, e^{h+\epsilon}))}{2h},$$

proves $\mathcal{D}^+(S) = \mathcal{D}^+(\tilde{S})$.

The claim concerning the lower density can be treated similarly. □

2.3. **Density of subsets of $\mathbb{R}$.** Since in this paper we are concerned with subsets of $\mathbb{R}$, we will state the definition of Beurling density only for this special case. Notice that the definition extends canonically to higher dimensions. For more details on the Beurling density in higher dimensions and its connections to Gabor frames we refer the reader [2], [3], and [13].

For a subset T of $\mathbb{R}$, the *upper Beurling density* of T is

$$\mathcal{D}^+(T) = \limsup_{h\to\infty} \frac{\sup_{x\in\mathbb{R}} \#(T \cap x + [-h, h))}{2h},$$

and the *lower Beurling density* of T is

$$\mathcal{D}^-(T) = \liminf_{h\to\infty} \frac{\inf_{x\in\mathbb{R}} \#(T \cap x + [-h, h))}{2h}.$$

If we have $\mathcal{D}^-(T) = \mathcal{D}^+(T)$, then T is said to possess the *uniform density* $\mathcal{D}(T)$.

2.4. **Connection between these densities.** Under some mild density conditions on $T \subset \mathbb{R}$, the density of $S \times T$ can be computed immediately from the densities of S and T.

Lemma 2.4. *Let $S \subset \mathbb{R}^+$ and $T \subset \mathbb{R}$. If T possesses a uniform density $\mathcal{D}(T)$, then $\mathcal{D}^-(S \times T) = \mathcal{D}^-(S)\,\mathcal{D}(T)$ and $\mathcal{D}^+(S \times T) = \mathcal{D}^+(S)\,\mathcal{D}(T)$.*

Proof. Fix $\epsilon > 0$. Since T possesses a uniform density, there exists $h_0 > 0$ with

$$\left| \frac{\#(T \cap x + [-h, h))}{2h} - \mathcal{D}(T) \right| < \epsilon \quad \text{for all } x \in \mathbb{R},\ h \ge h_0. \tag{4}$$

For each $(x, y) \in \mathbb{A}$,

$$\begin{aligned} \#((S \times T) \cap Q_h(x, y)) &= \#((S \times T) \cap \{(xa, \tfrac{y}{a} + b) : a \in [e^{-h}, e^h),\ b \in [-h, h)\}) \\ &= \sum_{s \in S \cap x[e^{-h}, e^h)} \#(b \in [-h, h) : \tfrac{xy}{s} + b \in T). \end{aligned}$$

Observing that

$$\#(b \in [-h, h) : \tfrac{xy}{s} + b \in T) = \#(T \cap \tfrac{xy}{s} + [-h, h)),$$

dividing by $4h^2$, taking the infimum over all $(x,y) \in \mathbb{A}$, and employing (4), yields

$$\begin{aligned}\inf_{x\in\mathbb{R}^+} \frac{\#(S\cap x[e^{-h},e^h))}{2h}\,(\mathcal{D}(T)-\epsilon) \;&\le\; \inf_{(x,y)\in\mathbb{A}} \frac{\#((S\times T)\cap Q_h(x,y))}{4h^2}\\ &\le\; \inf_{x\in\mathbb{R}^+} \frac{\#(S\cap x[e^{-h},e^h))}{2h}\,(\mathcal{D}(T)+\epsilon)\end{aligned}$$

for all $h \ge h_0$. Applying the liminf as $h \to \infty$ and noting that we can choose ϵ arbitrarily small, proves $\mathcal{D}^-(S\times T) = \mathcal{D}^-(S)\,\mathcal{D}(T)$.

The second claim can be treated similarly. □

2.5. **Notation for frames.** A system $\{f_i\}_{i\in I}$ in a separable Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$, if there exist $0 < A \le B < \infty$ (*lower and upper frame bounds*) such that $A\,\|f\|_2^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\,\|f\|_2^2$ for all $f \in \mathcal{H}$. If A, B can be chosen such that $A = B$, then $\{f_i\}_{i\in I}$ is a *tight frame*, and if we can take $A = B = 1$, it is called a *Parseval frame*.

3. Necessary density conditions for wavelet frames

3.1. **The fundamental relationship.** Given $r > 0$ and $T \subset \mathbb{R}$, we denote the corresponding sequence of exponentials by

$$\mathcal{E}(T,r) = \{x \mapsto e^{2\pi i t x} : t \in T, x \in [-r,r]\}.$$

Further, recall that a wavelet $\psi \in L^2(\mathbb{R})$ is called *admissible*, if $\int_{\mathbb{R}} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\,d\omega < \infty$.

The following result establishes a fundamental relationship between affine density, the frame bounds, and the admissibility constant for wavelet frames.

Theorem 3.1. *Let $\Lambda = S \times T \subset \mathbb{A}$, and let $\psi \in L^2(\mathbb{R})$ be admissible. If $\mathcal{W}(\psi,\Lambda)$ is a frame for $L^2(\mathbb{R})$ with frame bounds A and B, and $\mathcal{E}(T,r)$ is a frame for $L^2[-r,r]$ with frame bounds A_r and B_r for some $r > 0$, then*

$$\frac{A}{B_r} \le \mathcal{D}^-(S)\int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\,d\omega \le \mathcal{D}^+(S)\int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\,d\omega \le \frac{B}{A_r} \tag{5}$$

and

$$\frac{A}{B_r} \le \mathcal{D}^-(S)\int_{-\infty}^0 \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\,d\omega \le \mathcal{D}^+(S)\int_{-\infty}^0 \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\,d\omega \le \frac{B}{A_r}. \tag{6}$$

Moreover, if $\mathcal{E}(T,r)$ is a tight frame for $L^2[-r,r]$ for some $r > 0$, then

$$A \le \mathcal{D}^-(\Lambda)\int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\,d\omega \le \mathcal{D}^+(\Lambda)\int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\,d\omega \le B \tag{7}$$

and

$$A \le \mathcal{D}^-(\Lambda)\int_{-\infty}^0 \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\,d\omega \le \mathcal{D}^+(\Lambda)\int_{-\infty}^0 \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\,d\omega \le B. \tag{8}$$

Proof. We first show that the moreover-part follows from (5) and (6). It was proven in [9] that $\mathcal{E}(T, r)$ being a tight frame for $L^2[-r, r]$ for some $r > 0$ with frame bound C implies that T has uniform density $\mathcal{D}(T) = C$. Therefore, by Lemma 2.4, we obtain

$$\mathcal{D}^-(S) = \frac{\mathcal{D}^-(\Lambda)}{\mathcal{D}(T)} = \frac{\mathcal{D}^-(\Lambda)}{A_r} = \frac{\mathcal{D}^-(\Lambda)}{B_r}$$

and a similar result holds for the upper density. This shows that (7) and (8) follow from (5) and (6). Thus in the following we restrict our attention to the first two claims. In fact, we will only prove (5). The relation (6) can be treated similarly.

Now suppose $\mathcal{W}(\psi, \Lambda)$ is a frame for $L^2(\mathbb{R})$ with frame bounds A and B, and $\mathcal{E}(T, r)$ is a frame for $L^2[-r, r]$ with frame bounds A_r and B_r for some $r > 0$. Then [17, Thm. 1] implies that

$$\frac{A}{B_r} \leq \sum_{s\in S} |\hat{\psi}(s\omega)|^2 \leq \frac{B}{A_r} \quad \text{for a.e. } \omega \in \mathbb{R}^+. \tag{9}$$

For the sake of brevity, in this proof we will denote the boxes used in the definition of density for subsets of $\mathbb{R}^+$ by $K_h(x)$, i.e., $K_h = [e^{-h}, e^h)$ and $K_h(x) = xK_h$, where $h > 0$ and $x \in \mathbb{R}^+$. Let $\epsilon > 0$. Since ψ is admissible, we can choose $c > 0$ such that $\int_{\mathbb{R}^+\setminus K_c} \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega < \epsilon$. Further, fix $y \in \mathbb{R}^+$ and $h > c$. Dividing inequality (9) by ω and integrating each term in (9) over the box $K_h(y^{-1})$ yields

$$2h\frac{A}{B_r} \leq \sum_{s\in S} \int_{K_h(y^{-1})} \frac{|\hat{\psi}(s\omega)|^2}{\omega}\, d\omega \leq 2h\frac{B}{A_r}. \tag{10}$$

Then we make the decomposition

$$\sum_{s\in S} \int_{K_h(y^{-1})} \frac{|\hat{\psi}(s\omega)|^2}{\omega}\, d\omega = I_1(y,h) - I_2(y,h) + I_3(y,h) + I_4(y,h),$$

where

$$\begin{aligned}
I_1(y,h) &= \sum_{s\in S\cap K_{h-c}(y)} \int_0^\infty \frac{|\hat{\psi}(s\omega)|^2}{\omega}\, d\omega,\\
I_2(y,h) &= \sum_{s\in S\cap K_{h-c}(y)} \int_{\mathbb{R}^+\setminus K_h(y^{-1})} \frac{|\hat{\psi}(s\omega)|^2}{\omega}\, d\omega,\\
I_3(y,h) &= \sum_{s\in S\cap (K_{h+c}(y)\setminus K_{h-c}(y))} \int_{K_h(y^{-1})} \frac{|\hat{\psi}(s\omega)|^2}{\omega}\, d\omega,\\
I_4(y,h) &= \sum_{s\in S\cap (\mathbb{R}^+\setminus K_{h+c}(y))} \int_{K_h(y^{-1})} \frac{|\hat{\psi}(s\omega)|^2}{\omega}\, d\omega.
\end{aligned}$$

Since $\mathcal{W}(\psi, \Lambda)$ is a frame for $L^2(\mathbb{R})$, [16, Thm. 2.1 (1)] implies that $\mathcal{D}^+(S) < \infty$. By Lemma 2.1, there exists $N < \infty$ such that, for each $t > 0$, we have $\#(S \cap K_t(x)) \leq (t+1)\sup_{\tilde{x}\in\mathbb{R}^+} \#(S\cap K_1(\tilde{x})) \leq (t+1)N$ for all $x \in \mathbb{R}^+$. Notice that this immediately implies that also $\#(xS \cap K_t) \leq (t+1)N$ for all $x \in \mathbb{R}^+$.

We first observe that

$$I_1(y,h) = \#(S \cap K_{h-c}(y)) \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega.$$

To estimate $I_2(y,h)$, note that if $s \in K_{h-c}$, then $s(\mathbb{R}^+ \backslash K_h) = \mathbb{R}^+ \backslash K_h(s) \subseteq \mathbb{R}^+ \backslash K_c$. Therefore the contribution of $I_2(y,h)$ to the sum in (10) can be controlled by

$$\begin{aligned} I_2(y,h) &= \sum_{s \in y^{-1}S \cap K_{h-c}} \int_{\mathbb{R}^+ \backslash K_h} \frac{|\hat{\psi}(s\omega)|^2}{\omega}\, d\omega \\ &= \sum_{s \in y^{-1}S \cap K_{h-c}} \int_{\mathbb{R}^+ \backslash K_h(s)} \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega \\ &\le \#(y^{-1}S \cap K_{h-c}) \int_{\mathbb{R}^+ \backslash K_c} \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega \\ &\le (h-c+1)N\epsilon. \end{aligned}$$

Since $K_{h+c} \backslash K_{h-c}$ can be covered by a union of at most $2c+1$ intervals of the form $K_1(x)$, the term $I_3(y,h)$ can be estimated as follows:

$$I_3(y,h) = \sum_{s \in y^{-1}S \cap (K_{h+c} \backslash K_{h-c})} \int_{K_h} \frac{|\hat{\psi}(s\omega)|^2}{\omega}\, d\omega \le (2c+1)N \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega.$$

To estimate $I_4(y,h)$, note that if $s \notin K_{h+c}$, then $sK_h = K_h(s) \subseteq \mathbb{R}^+ \backslash K_c$. Furthermore, each interval in $\{K_h(s) : s \in S\}$ can intersect at most $h+1$ of the others. Hence the contribution of $I_4(y,h)$ can be controlled by

$$\begin{aligned} I_4(y,h) &= \sum_{s \in y^{-1}S \cap (\mathbb{R}^+ \backslash K_{h+c})} \int_{K_h} \frac{|\hat{\psi}(s\omega)|^2}{\omega}\, d\omega \\ &= \sum_{s \in y^{-1}S \cap (\mathbb{R}^+ \backslash K_{h+c})} \int_{K_h(s)} \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega \\ &\le (h+1)N \int_{\mathbb{R}^+ \backslash K_c} \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega \\ &\le (h+1)N\epsilon. \end{aligned}$$

Combining these estimates, we see that

$$2h\frac{A}{B_r} \le \#(S \cap K_{h-c}(y)) \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega + (h-c+1)N\epsilon + (2c+1)N \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega + (h+1)N\epsilon.$$

Therefore

$$
\begin{aligned}
\frac{A}{B_r} &= \liminf_{h\to\infty} \frac{2h\frac{A}{B_r}}{2h} \\
&\le \liminf_{h\to\infty} \inf_{x\in\mathbb{R}^+} \frac{\#(S\cap K_{h-c}(y))}{2h} \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega + \limsup_{h\to\infty} \frac{(h-c+1)N\epsilon}{2h} \\
&\quad + \limsup_{h\to\infty} \frac{(2c+1)N}{2h} \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega + \limsup_{h\to\infty} \frac{(h+1)N\epsilon}{2h} \\
&= \mathcal{D}^-(S) \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega + N\epsilon.
\end{aligned}
$$

Now letting ϵ go to zero yields $\frac{A}{B_r} \le \mathcal{D}^-(S) \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega$. The claim $\mathcal{D}^+(S) \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega \le \frac{B}{A_r}$ can be treated similarly. This settles (5). Hence the theorem is proved. $\square$

Remark 3.2. (a) In general the hypothesis that $\mathcal{E}(T,r)$ is a frame for $L^2[-r,r]$ for some $r>0$ is not restrictive, since it was shown by Jaffard in [11, Lem. 2] that $\mathcal{E}(T,r)$ is a frame for $L^2[-r,r]$ for some $r>0$ if and only if T is the disjoint union of a sequence with a uniform density and a finite number of uniformly discrete sequences, i.e., of sequences Δ which satisfy $\inf_{t_1,t_2\in\Delta, t_1\neq t_2} |t_1-t_2| > 0$. This is easily seen to be equivalent to $0 < \mathcal{D}^-(T) \le \mathcal{D}^+(T) < \infty$ (see, for instance, [9]).

However, $\mathcal{W}(\psi,\Lambda)$ being a frame for $L^2(\mathbb{R})$ does not imply $\mathcal{E}(T,r)$ being a frame for $L^2[-r,r]$ for some $r>0$. A counterexample for this fact was derived by Sun and Zhou in [16, Ex. 2.1].

(b) Consider the case $S = \{a^j : j\in\mathbb{Z}\}$, $a>1$ and $T = b\mathbb{Z}$, $b>0$. Then [10, Prop. 4.3] shows that $\mathcal{D}^-(S\times T) = \mathcal{D}^+(S\times T) = \frac{1}{b\ln a}$. Therefore Theorem 3.1 contains (1) as a special case.

3.2. **Some corollaries.** Theorem 3.1 yields several results interesting in their own right, which are all direct implications of this theorem.

It was stated as a conjecture in [10] and [15] that $\mathcal{W}(\psi,\Lambda)$ being a frame for $L^2(\mathbb{R})$ implies $\mathcal{D}^-(\Lambda) > 0$. Several partial results have already been discovered. The following corollary in fact generalizes the result from [16, Thm. 2.1 (2)], which states that $\mathcal{W}(\psi, S\times T)$ being a frame for $L^2(\mathbb{R})$ implies $\mathcal{D}^-(S) > 0$.

Corollary 3.3. *Let $\Lambda = S\times T \subset \mathbb{A}$, and let $\psi\in L^2(\mathbb{R})$ be admissible. If $\mathcal{W}(\psi,\Lambda)$ is a frame for $L^2(\mathbb{R})$, and $\mathcal{E}(T,r)$ is a tight frame for $L^2[-r,r]$ for some $r>0$, then $\mathcal{D}^-(\Lambda) > 0$.*

The second corollary shows that a wavelet system can only form a tight frame provided that the associated set of indices possesses a uniform affine density, thereby also delivering the exact value of the frame bound in terms of the affine density of the set of indices and the admissibility constant for the wavelet. This should be compared with the fact that the set of indices of classical wavelet systems always possesses a uniform affine density [10, Prop. 4.3]. The following result moreover does provide one reason, why there does not exist a Nyquist phenomenon for affine systems (see Subsection 3.3).

Corollary 3.4. *Let $\Lambda = S \times T \subset \mathbb{A}$, and let $\psi \in L^2(\mathbb{R})$ be admissible. If $\mathcal{W}(\psi, \Lambda)$ is a tight frame for $L^2(\mathbb{R})$ with frame bound A, and $\mathcal{E}(T, r)$ is a tight frame for $L^2[-r, r]$ for some $r > 0$, then Λ has uniform affine density and*

$$A = \mathcal{D}(\Lambda) \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega} \, d\omega = \mathcal{D}(\Lambda) \int_{-\infty}^0 \frac{|\hat{\psi}(\omega)|^2}{|\omega|} \, d\omega. \tag{11}$$

3.3. **The Nyquist phenomenon.** The following remark discusses the impact of Corollary 3.4 on the existence of a Nyquist density for affine systems.

Remark 3.5. We can view the necessary density condition (11) for $\mathcal{W}(\psi, \Lambda)$ to be a Parseval frame for $L^2(\mathbb{R})$, where $\mathcal{E}(T, r)$ is a tight frame for $L^2[-r, r]$ for some $r > 0$, also from the following perspective:

$$\mathcal{D}(\Lambda) = 2\|\hat{\psi}\|^{-2}_{L^2(\mathbb{R}, \frac{d\omega}{|\omega|})}. \tag{12}$$

Thus once $\mathcal{W}(\psi, \Lambda)$ constitutes an orthonormal basis for $L^2(\mathbb{R})$, and $\mathcal{E}(T, r)$ is a tight frame for $L^2[-r, r]$ for some $r > 0$, we obtain (12) as a necessary condition. The wavelet system being an orthonormal basis implies $\|\hat{\psi}\|_2 = 1$. However, we do not have any control over the constant $\|\hat{\psi}\|^{-2}_{L^2(\mathbb{R}, \frac{d\omega}{|\omega|})}$. Thus although Λ has a uniform affine density in this case, the value of it can range over the whole positive axis. In fact it can be shown that for each dilation factor $a > 1$, there exists a wavelet $\psi \in L^2(\mathbb{R})$ such that $\mathcal{W}(\psi, \{a^j : j \in \mathbb{Z}\} \times \mathbb{Z})$ is an orthonormal basis for $L^2(\mathbb{R})$ ([5, Ex. 4.5, Part 10]). Since $\mathcal{D}(\{a^j : j \in \mathbb{Z}\} \times \mathbb{Z}) = \frac{1}{\ln a}$ by [10, Prop. 4.3], the affine density can attain each positive value. Thus Corollary 3.4 reveals one reason, why wavelet systems do not possess a Nyquist density.

This consideration should be compared to recent results for Gabor systems [2], which show that if a Gabor system $\mathcal{G}(g, \Lambda)$, where $g \in L^2(\mathbb{R})$ and $\Lambda \subset \mathbb{R}^2$, forms an orthonormal basis for $L^2(\mathbb{R})$, it has to satisfy

$$\mathcal{D}(\Lambda) = \|\hat{g}\|_2^{-2}.$$

In this situation $\|\hat{g}\|_2 = 1$ immediately implies $\mathcal{D}(\Lambda) = 1$ in contrast to the wavelet systems, for which the norm of $\hat{\psi}$ needed for the computation of the uniform density is equipped with a different measure.

4. Sufficient density conditions for wavelet frames

Up to now density conditions have only served as necessary conditions. In this section we now show that density conditions can in fact be used to characterize the existence of wavelet frames. To prove this result we need the following technical lemma.

Lemma 4.1. *Let $S \subset \mathbb{R}^+$, and let f be in $L^1(\mathbb{R})$ with $f \geq 0$ and $f(x) \leq a|x|^\alpha$ as $|x| \to 0$ for some $a, \alpha > 0$. If $\mathcal{D}^+(S) < \infty$, then, for each $\epsilon > 0$, there exists $\gamma \in (0, 1)$ such that*

$$\sum_{s \in S} f(sx)\chi_{[0,\gamma)}(s|x|) < \epsilon \quad \text{for a.e. } x \in \mathbb{R}.$$

Proof. Fix $\epsilon > 0$, and let $\nu \in (0,1)$ be chosen so that $f(x) \leq a|x|^\alpha$ for almost every $|x| \leq \nu$. Since $\mathcal{D}^+(S) < \infty$, Lemma 2.1 shows the existence of some $N_{[\nu,1)} < \infty$ such that $\#(S \cap x[\nu,1)) \leq N_{[\nu,1)}$ for all $x \in \mathbb{R}^+$. Then, for each $n \in \mathbb{N}$ and almost every $x \in \mathbb{R}$,

$$\begin{aligned}\sum_{s\in S} f(sx)\chi_{[0,\nu^n)}(s|x|) &\leq a \sum_{s\in S\cap[0,|x|^{-1}\nu^n)} (s|x|)^\alpha \\ &= a|x|^\alpha \sum_{j=n}^{\infty} \sum_{s\in S\cap|x|^{-1}[\nu^{j+1},\nu^j)} s^\alpha \\ &\leq a|x|^\alpha \sum_{j=n}^{\infty} N_{[\nu,1)}(|x|^{-1}\nu^j)^\alpha \\ &= aN_{[\nu,1)} \sum_{j=n}^{\infty} (\nu^\alpha)^j,\end{aligned}$$

which is finite. Thus we can choose $n_0 \in \mathbb{N}$ such that

$$\sum_{s\in S} f(sx)\chi_{[0,\nu^{n_0})}(s|x|) < \epsilon.$$

Setting $\gamma = \nu^{n_0}$ settles the claim. □

The next result shows that the existence of frames of band-limited admissible wavelets with a certain decay condition can be characterized by using a condition on the density of the set of dilations.

Theorem 4.2. *Let $S \subset \mathbb{R}^+$, and let ψ be in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and admissible with $|\hat{\psi}(\omega)| \leq a|\omega|^\alpha$ as $|\omega| \to 0$ for some $a, \alpha > 0$, where $\omega = 0$ is an isolated zero of $\hat{\psi}$, and $|\hat{\psi}(\omega)| = 0$ for any $|\omega| \geq \Omega$. Then the following conditions are equivalent.*

(i) *There exists $T \subset \mathbb{R}$ such that $\mathcal{E}(T,r)$ is a frame for $L^2[-r,r]$, where $r > 2\Omega$, and $\mathcal{W}(\psi, S \times T)$ is a frame for $L^2(\mathbb{R})$.*

(ii) *$0 < \mathcal{D}^-(S) \leq \mathcal{D}^+(S) < \infty$.*

Moreover, if (ii) holds, then $\mathcal{W}(\psi, S \times T)$ is a frame for $L^2(\mathbb{R})$ for any $T \subset \mathbb{R}$ satisfying that $\mathcal{E}(T,r)$ constitutes a frame for $L^2[-r,r]$, where $r > 2\Omega$.

Proof. The implication (i) ⇒ (ii) follows immediately from Theorem 3.1.

Now suppose (ii) holds. First we will prove that $\mathcal{D}^+(S) < \infty$ implies the existence of some $B < \infty$ satisfying

$$\sum_{s\in S} |\hat{\psi}(s\omega)|^2 \leq B \quad \text{for all } \omega \in \mathbb{R}. \tag{13}$$

Employing Lemma 4.1 shows that for some $\epsilon > 0$ there exists $0 < \gamma < 1$ such that

$$\sum_{s\in S} |\hat{\psi}(s\omega)|^2 \chi_{[0,\gamma)}(s|\omega|) < \epsilon \quad \text{for all } \omega \in \mathbb{R}.$$

We now focus on the sum $\sum_{s\in S} |\hat{\psi}(s\omega)|^2 \chi_{[\gamma,\Omega)}(s|\omega|)$. Our hypotheses imply that there exists $M > 0$ satisfying $|\omega||\hat{\psi}(\omega)| \leq M$ for each $\omega \in \mathbb{R}$. By Lemma 2.1, we have $\#(S \cap \omega[\gamma,\Omega)) \leq$

$N_{[\gamma,\Omega)} < \infty$ for all $\omega \in \mathbb{R}^+$. Hence, for each $\omega \in \mathbb{R}$, we get

$$\begin{aligned}\sum_{s\in S} |\hat{\psi}(s\omega)|^2 \chi_{[\gamma,\Omega)}(s|\omega|) &\leq \sum_{s\in S\cap|\omega|^{-1}[\gamma,\Omega)} M^2 s^{-2}|\omega|^{-2} \\ &\leq M^2(|\omega|^{-1}\gamma)^{-2} N_{[\gamma,\Omega)}|\omega|^{-2} \\ &= M^2 N_{[\gamma,\Omega)}\gamma^{-2}.\end{aligned}$$

This settles (13).

Secondly, we employ the hypothesis $\mathcal{D}^-(S) > 0$. We claim that this implies that there exists $A > 0$ such that

$$\sum_{s\in S} |\hat{\psi}(s\omega)|^2 \geq A \quad \text{for all } \omega \in \mathbb{R}. \tag{14}$$

Since $\omega = 0$ is an isolated zero of $\hat{\psi}$, we can choose $\epsilon > 0$ with $\hat{\psi}(\omega) \neq 0$ for each $\omega \in (0, \epsilon)$. Now Lemma 2.2 implies the existence of some interval $I \subset \mathbb{R}^+$ of positive finite measure and some positive constant N_I satisfying

$$\#(S \cap \omega I) > N_I \quad \text{for all } \omega \in \mathbb{R}^+. \tag{15}$$

Let $\omega_0 \in \mathbb{R}^+$ be chosen so that $\omega_0 I \subset (0, \epsilon)$. Since $\hat{\psi}$ is continuous, we have $|\hat{\psi}(\omega)| \geq \delta$ on $\omega_0 I$ for some $\delta > 0$. Now fix some $\omega \in \mathbb{R}^+$. Then (15) implies the existence of some $s_0 \in S$ such that $s_0 \in \omega^{-1}\omega_0 I$. This immediately yields

$$\sum_{s\in S} |\hat{\psi}(s\omega)|^2 \geq |\hat{\psi}(s_0\omega)|^2 \geq \delta^2,$$

thereby proving (14). Now the implication (ii) $\Rightarrow$ (i) and the moreover-part follows from (13) and (14) and [17, Cor. 1]. □

Remark 4.3. We point out that a related result on sufficient conditions for irregular (weighted) wavelet frames was derived by Gröchenig in [8]. To emphasize the difference to our Theorem 4.2, we observe that the focus in [8, Thm. 1] is on the introduction of adaptive weights to compensate for local variations of the set of indices, thereby deriving a *weighted* wavelet frame. The two results are distinct and complementary.

Next we briefly remark on whether it is possible to weaken the hypotheses of the previous proposition and on a possible improvement.

Remark 4.4. (a) If ω is not an isolated zero of $\hat{\psi}$, it is easy to check that the implication (ii) $\Rightarrow$ (i) does not automatically hold. For instance, if we let $S = \{2^j : j \in \mathbb{Z}\}$, for which $\mathcal{D}^-(S) = \mathcal{D}^+(S) = \frac{1}{\ln 2}$, and define $\psi \in L^2(\mathbb{R})$ by $\hat{\psi} = \chi_{[1,\frac{3}{2})}$, then $\mathcal{W}(\psi, S \times T)$ is not even complete, since $\bigcup_{j\in\mathbb{Z}} 2^j[1, \frac{3}{2})$ does not cover $\mathbb{R}$. Thus it follows that there does not exist a frame $\mathcal{E}(T, r)$, $T \subset \mathbb{R}$, for $L^2[-r, r]$ for any $r > 0$ such that $\mathcal{W}(\psi, S\times T)$ is a frame for $L^2(\mathbb{R})$.

(b) One might further ask, whether it is possible to include the values of the frame bounds of the frame from (i) in condition (ii). However, it is not too difficult to see that this is not possible. One reason is that in fact there exists an abundance of possibilities for choosing $\mathcal{E}(T, r)$ with different frame bounds as indicated by the moreover-part of Theorem 4.2, thus changing the frame bounds of $\mathcal{W}(\psi, S \times T)$ while S remains the same.

Now the question arises, whether it is also sufficient to consider density conditions concerning the existence of Parseval frames, i.e., whether we can obtain a similar equivalence in the situation of Parseval frames. However, the following result shows that this would be too much to hope for. One reason for this is that Parseval frames are very sensitive to perturbations of the indices, but density is not (see Lemma 2.3).

Proposition 4.5. *For any $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $|\hat{\psi}(\omega)| \le a|\omega|^\alpha$ as $|\omega| \to 0$ for some $a, \alpha > 0$, where $\omega = 0$ is an isolated zero of $\hat{\psi}$, and $|\hat{\psi}(\omega)| = 0$ for any $|\omega| \ge \Omega$, and for any $T \subset \mathbb{R}$ satisfying that $\mathcal{E}(T,r)$ is a frame for $L^2[-r,r]$, where $r > 2\Omega$, there exists $S \subset \mathbb{R}^+$ with positive finite uniform density such that $\mathcal{W}(\psi, S \times T)$ does not form a Parseval frame for $L^2(\mathbb{R})$.*

Proof. Let $\psi \in L^2(\mathbb{R})$ and $T \subset \mathbb{R}$ be chosen such that they satisfy the hypotheses of the proposition. Let A_r and B_r denote the frame bounds of $\mathcal{E}(T,r)$ in $L^2[-r,r]$. By the hypotheses, $\hat{\psi}$ is continuous, hence there exists an interval $I \subset \mathbb{R}$ and $\delta > 0$ such that $|\hat{\psi}(\omega)| \ge \delta$ for all $\omega \in I$. Without loss of generality we can assume that $I \subset \mathbb{R}^+$ and that there exists $j_0 \ge 2$ so that I is a proper subset of $(\frac{\Omega}{2^{j_0-1}}, \frac{\Omega}{2^{j_0-2}}]$. Let $0 < \epsilon < \frac{1}{B_r}$. Setting $m := \lceil (\frac{1}{A_r} - \frac{1}{B_r} + \epsilon)/\delta^2 \rceil + 1$, we can choose m disjoint elements a_k, $1 \le k \le m$ such that there exists $U \subset (\frac{\Omega}{2^{j_0+1}}, \frac{\Omega}{2^{j_0}}]$ of positive measure satisfying that $a_k U \subset I$ for all $1 \le k \le m$. In particular, this implies that $a_k > 2$ for $1 \le k \le m$. Now define $S \subset \mathbb{R}^+$ by $S := \{2^j\}_{j \in \mathbb{Z}} \cup \{a_k\}_{k=1}^m$. An easy computation shows that S has a positive finite uniform density equal to $\frac{1}{\ln 2}$. By [17, Thm. 1], it suffices to show that provided

$$\sum_{s \in S} |\hat{\psi}(s\omega)|^2 \ge \frac{1}{B_r} \quad \text{for all } \omega \in \mathbb{R}^+,$$

there exists a set $U \subset \mathbb{R}$ of positive measure with

$$\sum_{s \in S} |\hat{\psi}(s\omega)|^2 > \frac{1}{A_r} \quad \text{for all } \omega \in U.$$

Lemma 4.1 proves that there exists some $0 < \gamma < 1$ with

$$\sum_{s \in S} |\hat{\psi}(s\omega)|^2 \chi_{[0,\gamma)}(s\omega) < \epsilon \quad \text{for all } \omega \in \mathbb{R}^+.$$

Noting that we can assume that $\gamma = 2^{-j_1}\Omega$ for some $j_1 \in \mathbb{Z}$ by choosing γ slightly smaller if necessary, we obtain that, for all $\omega \in (\frac{\Omega}{2}, \Omega]$,

$$\frac{1}{B_r} \le \sum_{s \in S} |\hat{\psi}(s\omega)|^2 = \sum_{j=-\infty}^{0} |\hat{\psi}(2^j\omega)|^2 \le \sum_{j=-j_1}^{0} |\hat{\psi}(2^j\omega)|^2 + \epsilon. \tag{16}$$

Now let $\omega \in U$. Using (16), we compute

$$\begin{aligned}\sum_{s\in S} |\hat{\psi}(s\omega)|^2 &= \sum_{j=-\infty}^{j_0} |\hat{\psi}(2^j\omega)|^2 + \sum_{k=1}^{m} |\hat{\psi}(a_k\omega)|^2 \\ &\geq \sum_{j=j_0-j_1}^{j_0} |\hat{\psi}(2^j\omega)|^2 + \sum_{k=1}^{m} |\hat{\psi}(a_k\omega)|^2 \\ &= \sum_{j=-j_1}^{0} |\hat{\psi}(2^j 2^{j_0}\omega)|^2 + \sum_{k=1}^{m} |\hat{\psi}(a_k\omega)|^2 \\ &\geq \frac{1}{B_r} - \epsilon + m\delta^2 > \frac{1}{A_r}.\end{aligned}$$

Hence the proposition is proved. □

Notice that once there exists one frame $\mathcal{E}(T, r)$ for $L^2[-r, r]$, where $T \subset \mathbb{R}$ and $r > 2\Omega$, such that $\mathcal{W}(\psi, S \times T)$ is a frame for $L^2(\mathbb{R})$, the moreover-part of Theorem 4.2 implies that $\mathcal{W}(\psi, S \times T)$ is a frame for any such system $\mathcal{E}(T, r)$. Thus the previous result indeed proved that we cannot expect a similar equivalence as in Theorem 4.2 for Parseval frames.

Acknowledgments

The research for this paper was performed while the author was visiting the Departments of Mathematics at Washington University in St. Louis and the Georgia Institute of Technology. We thank these departments for their hospitality and support during these visits.

We are also indebted to Akram Aldroubi, Christopher Heil, Mihail Kolountzakis, Demetrio Labate, Amos Ron, Zouwei Shen, Guido Weiss, and Edward Wilson for helpful discussions. Moreover, we thank the anonymous referees for suggesting the reference [8] and for other helpful comments.

The author acknowledges support from DFG research fellowship KU 1446/5.

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