The theory of reproducing systems on locally compact abelian groups

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Abstract

A reproducing system is a countable collection of functions $\{\phi_j : j \in \mathcal{J}\}$ such that a general function f can be decomposed as $f = \sum_{j\in\mathcal{J}} c_j(f)\,\phi_j$, with some control on the analyzing coefficients $c_j(f)$. Several such systems have been introduced very successfully in mathematics and its applications. We present a unified viewpoint to the study of reproducing systems on locally compact abelian groups G. This approach gives a novel characterization of the Parseval frame generators for a very general class of reproducing systems on $L^2(G)$. As an application of this result, we obtain a new characterization of Parseval frame generators for Gabor and affine systems on $L^2(G)$.

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1 Introduction

The term **reproducing system** is applied to any of several methods that decompose a general function f in terms of a countable system of functions

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$\{\phi_j : j \in \mathcal{J}\}$ so that

$$f = \sum_{j \in \mathcal{J}} c_j(f)\, \phi_j,$$

where the $c_j(f)$ are appropriate coefficient functionals, and the norm of f is equivalent to the norm of the coefficients $\{c_j(f) : j \in \mathcal{J}\}$. A variety of such systems has been used very successfully in both pure and applied mathematics. They are generated by a single or a finite collection of functions, by applying to these functions a countable family of operators. These operators involve typically two of the following three actions: dilations, modulations, and translations. The **Gabor systems**, for example, have the form

$$\mathcal{G}_B(\Psi) = \{M_{Bm}\, T_k\, \psi^\ell : \; m, k \in \mathbb{Z}^n, \ell = 1, \dots, L\}, \tag{1}$$

where $\Psi = (\psi^1, \dots, \psi^L) \subset L^2(\mathbb{R}^n)$, $B \in GL_n(\mathbb{R})$, T_k are the **translations**, defined by $T_k\, f(x) = f(x-k)$, and M_y are the **modulations**, defined by $M_y\, f(x) = e^{2\pi i \langle y,x\rangle}\, f(x)$. The **affine systems** (which generate a variety of wavelets), on the other hand, have the form

$$\mathcal{W}_A(\Psi) = \{D_{A^j}\, T_k\, \psi^\ell : \; j \in \mathbb{Z}, k \in \mathbb{Z}^n, \ell = 1, \dots, L\}, \tag{2}$$

where $A \in GL_n(\mathbb{R})$ and D_A are the **dilations**, that are defined by $D_A\, f(x) = |\det A|^{-1/2}\, f(A^{-1}x)$. By choosing Ψ, B, and A appropriately, one can make $\mathcal{G}_B(\Psi)$ and $\mathcal{W}_A(\Psi)$ an orthonormal basis or, more generally, a Parseval frame for $L^2(\mathbb{R}^n)$ (defined below).

While the theory of Gabor and affine systems has usually been investigated on $\mathbb{R}^n$, there is an increasing interest in the study of these systems in other settings (for example, [1,7,12,16]). Indeed, discrete signal processing applications, as well as numerical implementations of these theories, require the construction of reproducing systems on $\mathbb{Z}^n$ or finite abelian groups. Moreover, in several applied problems, as for example in the numerical solution of PDE, one has to deal with bounded domains, and the Gabor or affine systems on $\mathbb{R}^n$ are unable to handle effectively the boundary conditions. Therefore it is quite useful to consider reproducing systems adapted to bounded domains.

One of the aims of this paper is to extend many ideas and constructions from the theory of Gabor and affine systems to the setting of locally compact abelian groups. One major result is a novel characterization of all those functions that form a Parseval frame for $L^2(G)$, where G is a locally compact abelian group. This allows us to handle several classes of reproducing systems on $L^2(G)$ in a unified manner. As an application of this approach, we are able to extend and generalize several results from the theory of Gabor and affine systems on $\mathbb{R}^n$ to the setting of general locally compact abelian groups.

The paper is organized as follows. We recall some basic facts about locally compact abelian groups and frame theory in Section 2. In Section 3 we present our general characterization result for Parseval frame generators. Finally in Section 4 we describe several applications of this theorem, including the cases of Gabor and affine systems.

2 Preliminaries

Before embarking on our study, it is useful to establish some notation and recall some basic facts from the theory of locally compact abelian groups. More details can be found, for example, in the monographs [15,20].

Let G denote a locally compact abelian group with unit element e. G is equipped with a left-invariant Haar measure m_G, which is unique up to a constant multiple, and is finite if and only if G is compact. In addition, we will assume that G is σ-compact, i.e., G is a countable union of compact sets, and metrizable, i.e., there is a metric d on G. Any locally compact metrizable σ-compact abelian group will be called a **LCA group**.

The dual group of G, that we denoted by $\widehat{G}$, is the set of all **characters**, i.e., all continuous homomorphisms from G into the torus $\mathbb{T}$. It turns out that $\widehat{G}$ also becomes a LCA group under the pointwise multiplication, with unit element 1, and thus possesses a Haar measure.

A discrete subgroup D of G with compact quotient group G/D will be called a **uniform lattice**. A **fundamental domain** for D is a measurable subset $F \subset G$ such that every $x \in G$ can be uniquely written in the form $x = fd$ for some $f \in F$ and $d \in D$. It was shown in [18, Lemma 2] that such a fundamental domain always exists. We define the **lattice size** of D to be $s(D) = m_G(F)$. It can be easily shown that this definition is independent of the particular choice of F. The **annihilator** of D in G, denoted by $D^\perp$, is defined by

$$D^\perp = \{\gamma \in \widehat{G} : \gamma(d) = 1 \text{ for all } d \in D\}.$$

Then $D^\perp$ is a uniform lattice in $\widehat{G}$, since $D^\perp$ is topologically isomorphic to $\widehat{G/D}$ and $\widehat{G}/D^\perp$ is topologically isomorphic to $\widehat{D}$ (via the restriction map $\omega D^\perp \to \omega|D$). The following lemma [12, Lemma 6.2.3(a)] will be useful.

Lemma 1 *If D is a uniform lattice in G, then we have*

$$s(D)s(D^\perp) = 1.$$

As usually, $L^2(G)$ is the space of square-integrable functions on G with respect to m_G, $L^1(G)$ is the space of integrable functions on G. Note that in the following we will just write $\int_G f(x)\,dx$ rather than $\int_G f(x)\,dm_G(x)$, and will always assume the Haar measure on the compact group G/D to be normalized.

Let D be a uniform lattice in G and let F be an associated fundamental domain. Choosing the counting measure on D, a relation between the Haar measures on G and G/D is given by the following special case of Weil's formula [20]. For $f \in L^1(G)$, we have that $\sum_{d\in D} f(xd) \in L^1(G/D)$ and

$$\int_G f(x)\,dm_G(x) = s(D) \int_{G/D} \left(\sum_{d\in D} f(xd)\right) dm_{G/D}(\dot{x}), \tag{3}$$

where $\dot{x} = xD$ (later on, if the context is clear, we will write simply $d\dot{x}$ rather than $dm_{G/D}(\dot{x})$). The Fourier transform $\hat{f}$ of any function $f \in L^1(G)$ is defined by

$$\hat{f}(\omega) = \int_G f(t)\,\overline{\omega(t)}\,dm_G(t).$$

The transformation $f \mapsto \hat{f}$, $L^1(G) \to C_0(\widehat{G})$ extends to a Hilbert space isomorphism of $L^2(G)$ onto $L^2(\widehat{G})$, the so-called **Plancherel isomorphism**. In the sequel, the Plancherel transform of a function $f \in L^2(G)$ will also be denoted by $\hat{f}$. Throughout this paper, we will always assume that the Haar measure on $\widehat{G}$, μ_G, is normalized so that the Plancherel formula holds, i.e., we have

$$\int_G |f(x)|^2\,dm_G(x) = \int_{\widehat{G}} |\hat{f}(\omega)|^2\,d\mu_G(\omega),$$

for any $f \in L^2(G)$.

The following definitions will also be needed.

A countable family $\{e_j : j \in \mathcal{J}\}$ of elements in a separable Hilbert space $\mathcal{H}$ (for example, $\mathcal{H} = L^2(G)$, where G is a LCA group) is a **frame** if there exist constants $0 < \alpha \le \beta < \infty$ satisfying

$$\alpha\,\|v\|^2 \;\le\; \sum_{j\in\mathcal{J}} |\,\langle v, e_j\rangle\,|^2 \;\le\; \beta\,\|v\|^2$$

for all $v \in \mathcal{H}$. If the right hand side inequality, but not necessarily the left hand side one holds, we say that $\{e_j : j \in \mathcal{J}\}$ is a **Bessel system** with constant β. A frame is **tight** if α and β can be chosen so that $\alpha = \beta$, and is a **Parseval**

frame if $\alpha = \beta = 1$. Thus, if $\{e_j : j \in \mathcal{J}\}$ is a Parseval frame in $\mathcal{H}$, then

$$\|v\|^2 = \sum_{j \in \mathcal{J}} | \langle v, e_j \rangle |^2$$

for each $v \in \mathcal{H}$. This is equivalent to the reproducing formula

$$v = \sum_{j \in \mathcal{J}} \langle v, e_j \rangle \; e_j \tag{4}$$

for all $v \in \mathcal{H}$, where the series in (4) converges in the norm of $\mathcal{H}$. We refer the reader to [9,4] for the basic properties of frames.

3 Characterization of Parseval frame generators

It is well-known that there are relatively simple equations that characterize those functions $\Psi = \{\psi^1, \ldots, \psi^L\}$ for which a Gabor system $\mathcal{G}_B(\Psi)$, given by (1), or an affine system $\mathcal{W}_A(\Psi)$, given by (2), is a Parseval frame for $L^2(\mathbb{R}^n)$. Several papers have been devoted to the formulation and the study of these characterizations, and they play a major role in the construction and the study of Gabor and affine systems (for example, [2,3,5,10,11,17,13,14,19,21–23]). The approach that we develop in this paper adapts some ideas from [13,19], where one of the authors of this paper has developed a unified approach to the study of Gabor systems and affine systems in $L^2(\mathbb{R}^n)$.

Let G be a LCA group, $\mathcal{P}$ a countable index set, $\{g_p : p \in \mathcal{P}\}$ a family of functions in $L^2(G)$, and $\{D_p : p \in \mathcal{P}\}$ a collection of uniform lattices in G. For $x \in G$, let the translation operator T_x on $L^2(G)$ be defined by $T_x f(t) = f(tx^{-1})$. We will consider families of the form

$$\Phi^{\{g_p\}}_{\{D_p\}} \; = \; \{T_{\lambda_p} \, g_p : \lambda_p \in D_p, \, p \in \mathcal{P}\}. \tag{5}$$

In order to state our general characterization result, we introduce the following notation. Let $\Lambda = \bigcup_{p \in \mathcal{P}} D_p^{\perp}$, and, for each $\alpha \in \Lambda$, let $\mathcal{P}_\alpha = \{p \in \mathcal{P} : \alpha \in D_p^{\perp}\}$. We also need the following definition.

Definition 2 The system (5) satisfies the **local integrability condition (LIC)**, if, for each compact subset K of $\widehat{G}$, we have

$$\sum_{p \in \mathcal{P}} s(D_p)^{-1} \sum_{\gamma_p \in D_p^{\perp}} \Big(\int_{K \cap \gamma_p^{-1} K} |\hat{g}_p(\omega)|^2 d\omega \Big) < \infty. \tag{6}$$

Let

$$\mathcal{D} = \{f \in L^2(G) : \hat{f} \in L^\infty(\widehat{G}) \text{ and } \operatorname{supp} \hat{f} \text{ is compact}\}. \tag{7}$$

Observe that $\mathcal{D}$ is a dense subset of $L^2(G)$. If the system (5) satisfies the LIC, then it is clear that, for each $f \in \mathcal{D}$, we have

$$\sum_{p\in\mathcal{P}} \sum_{\gamma_p \in D_p^\perp} \Big(\int_{\operatorname{supp} \hat{f}} |\hat{f}(\omega\gamma_p)|^2 s(D_p)^{-1} |\hat{g}_p(\omega)|^2 d\omega \Big) < \infty. \tag{8}$$

Indeed, one can show that this statement is equivalent to the LIC.

We can now state our general characterization result.

Theorem 3 *Let $\mathcal{P}$ be a countable index set, $\{g_p : p \in \mathcal{P}\}$ a family of functions in $L^2(G)$, and $\{D_p : p \in \mathcal{P}\}$ a collection of uniform lattices in G. Suppose that the set $\Phi_{\{D_p\}}^{\{g_p\}}$, given by (5), satisfies the LIC. Then the following conditions are equivalent.*

(i) $\Phi_{\{D_p\}}^{\{g_p\}}$ *is a Parseval frame for $L^2(G)$.*
(ii) *For each $\alpha \in \Lambda$, we have*

$$\sum_{p\in\mathcal{P}_\alpha} s(D_p)^{-1} \overline{\hat{g}_p(\omega)} \hat{g}_p(\omega\alpha) = \delta_{\alpha,1} \quad \textit{for a.e. } \omega \in \widehat{G}. \tag{9}$$

This general result will be later applied to several special families of functions $\Phi_{\{D_p\}}^{\{g_p\}}$. Observe that in many of these cases we will be able to remove the LIC hypothesis from the corresponding characterization therorem. The proof of Theorem 3 will adapt some ideas from the proof of [13, Theorem 2.1]. Before presenting this proof, we need to establish the followng lemmas.

Let D be a uniform lattice in G and let $f, g \in L^2(G)$. The D**-bracket product** of f and g, which was originally introduced in [8] and extended in [13], is defined, in our setting, as

$$[f, g](x; D) = \sum_{d\in D} f(xd)\, \overline{g(xd)}. \tag{10}$$

The function $[f, g](x; D)$ in (10) is also called the **periodization of f and g with respect to D**. In the following, we establish the following useful lemmas.

Lemma 4 *Let D be a uniform lattice. If $f \in \mathcal{D}$, where $\mathcal{D}$ is given by (7), and*

$g \in L^2(G)$, *then*

$$\sum_{d\in D} |\langle f, T_d\, g\rangle|^2 = s(D)^{-2} \int\limits_{\widehat{G}/D^\perp} |[\hat{f}, \hat{g}](\omega; D^\perp)|^2 \, d\dot{\omega}.$$

PROOF. Since $(T_d\, g)\hat{}\,(\omega) = \overline{\omega(d)}\, \hat{g}(\omega)$, the Plancherel theorem implies that

$$\sum_{d\in D} |\langle f, T_d\, g\rangle|^2 = \sum_{d\in D} \left| \int\limits_{\widehat{G}} \hat{f}(\omega)\, \overline{\hat{g}(\omega)}\, \omega(d)\, d\omega \right|^2 .$$

By applying Weil's formula (3) and Lemma 1, we obtain

$$\begin{aligned} \int\limits_{\widehat{G}} \hat{f}(\omega)\, \overline{\hat{g}(\omega)}\, \omega(d)\, d\omega &= s(D^\perp) \int\limits_{\widehat{G}/D^\perp} \left(\sum_{\gamma\in D^\perp} \hat{f}(\omega\gamma)\, \overline{\hat{g}(\omega\gamma)}\, (\omega\gamma)(d) \right) d\dot{\omega} \\ &= s(D)^{-1} \int\limits_{\widehat{G}/D^\perp} [\hat{f}, \hat{g}](\omega; D^\perp)\, \omega(d)\, d\dot{\omega}. \end{aligned}$$

Observe that $\widehat{G}/D^\perp$ is topologically isomorphic to $\widehat{D}$. Thus, by choosing the Haar measure on $\widehat{D}$ (via this isomorphism) and using once more the Plancherel theorem, we obtain

$$\begin{aligned} \sum_{d\in D} \left| \int\limits_{\widehat{G}} \hat{f}(\omega)\, \overline{\hat{g}(\omega)}\, \omega(d)\, d\omega \right|^2 &= \sum_{d\in D} \left| s(D)^{-1} \int\limits_{\widehat{G}/D^\perp} [\hat{f}, \hat{g}](\omega; D^\perp)\, \omega(d)\, d\dot{\omega} \right|^2 \\ &= s(D)^{-2} \sum_{d\in D} \left| \int\limits_{\widehat{D}} [\hat{f}, \hat{g}](\omega; D^\perp)\, \omega(d)\, d\omega \right|^2 \\ &= s(D)^{-2} \int\limits_{\widehat{D}} |[\hat{f}, \hat{g}](\omega; D^\perp)|^2 \, d\omega \\ &= s(D)^{-2} \int\limits_{\widehat{G}/D^\perp} |[\hat{f}, \hat{g}](\omega; D^\perp)|^2 \, d\dot{\omega}. \qquad \square \end{aligned}$$

Lemma 5 *Let D be a uniform lattice in G. For each $f \in \mathcal{D}$ and $g \in L^2(G)$, define the function H on G by*

$$H(x) = \sum_{d\in D} |\langle T_x\, f, T_d\, g\rangle|^2 .$$

Then $H : G/D \mapsto \mathbb{R}$ is the trigonometric polynomial:

$$H(x) = \sum_{\gamma \in D^\perp} \left(s(D)^{-1} \int_{\widehat{G}} \hat{f}(\omega)\, \overline{\hat{f}(\omega\gamma)}\, \overline{\hat{g}(\omega)}\, \hat{g}(\omega\gamma)\, d\omega \right) \gamma(x).$$

PROOF. By Lemma 4, we have

$$\begin{aligned} s(D)^2 H(x) &= \int_{\widehat{G}/D^\perp} |[(T_x f)\hat{}, \hat{g}](\omega; D^\perp)|^2\, d\dot{\omega} \\ &= \int_{\widehat{G}/D^\perp} \left| \overline{\omega(x)} \sum_{\gamma \in D^\perp} \overline{\gamma(x)}\, \hat{f}(\omega\gamma)\, \overline{\hat{g}(\omega\gamma)} \right|^2 d\dot{\omega} \\ &= \int_{\widehat{G}/D^\perp} \sum_{\gamma \in D^\perp} \overline{\gamma(x)}\, \hat{f}(\omega\gamma)\, \overline{\hat{g}(\omega\gamma)} \sum_{\delta \in D^\perp} \delta(x)\, \overline{\hat{f}(\omega\delta)}\, \hat{g}(\omega\delta)\, d\dot{\omega}. \end{aligned}$$

Next, we use the substitution $\delta = \gamma\eta$ and express the last integrand as a sum over γ and η; then, by applying Weil's formula and Lemma 1, we obtain

$$\begin{aligned} s(D)^2 H(x) &= \sum_{\gamma \in D^\perp} \int_{\widehat{G}/D^\perp} \hat{f}(\omega\gamma)\, \overline{\hat{g}(\omega\gamma)} \sum_{\eta \in D^\perp} \eta(x)\, \overline{\hat{f}(\omega\gamma\eta)}\, \hat{g}(\omega\gamma\eta)\, d\omega \\ &= s(D^\perp)^{-1} \int_{\widehat{G}} \hat{f}(\omega)\, \overline{\hat{g}(\omega)} \sum_{\eta \in D^\perp} \eta(x)\, \overline{\hat{f}(\omega\eta)}\, \hat{g}(\omega\eta)\, d\omega \\ &= s(D) \sum_{\eta \in D^\perp} \left(\int_{\widehat{G}} \hat{f}(\omega)\, \overline{\hat{f}(\omega\eta)}\, \overline{\hat{g}(\omega)}\, \hat{g}(\omega\eta)\, d\omega \right) \eta(x). \end{aligned}$$

Notice that all exchanges in the order of summations and integrations are justified since $f \in \mathcal{D}$. $\square$

The following result will be the main tool in the proof of Theorem 3.

Proposition 6 *Let $\mathcal{P}$ be a countable index set, $\{g_p : p \in \mathcal{P}\}$ a family of functions in $L^2(G)$, and $\{D_p : p \in \mathcal{P}\}$ a collection of uniform lattices in G. Suppose that the collection $\Phi^{\{g_p\}}_{\{D_p\}}$, given by (5), satisfies the LIC. For each $f \in \mathcal{D}$, define the functional N^2 on $L^2(G)$ by*

$$N^2(f) = \sum_{p \in \mathcal{P}} \sum_{\lambda_p \in D_p} \left| \left\langle f, T_{\lambda_p} g_p \right\rangle \right|^2.$$

Then the function $w(x) = N^2(T_x f)$ is continuous and coincides pointwise with its absolutely convergent Fourier series $\sum_{\alpha \in \Lambda} \hat{w}(\alpha)\, \alpha(x)$, where

$$\hat{w}(\alpha) = \int_{\widehat{G}} \hat{f}(\omega)\, \overline{\hat{f}(\omega\alpha)} \sum_{p \in \mathcal{P}_\alpha} s(D_p)^{-1}\, \overline{\hat{g}_p(\omega)}\, \hat{g}_p(\omega\alpha)\, d\omega, \tag{11}$$

and the last integral converges absolutely.

PROOF. We have

$$w(x) = \sum_{p \in \mathcal{P}} \sum_{\lambda_p \in D_p} \left| \left\langle f, T_{\lambda_p x^{-1}}\, g_p \right\rangle \right|^2 .$$

Now for each $p \in \mathcal{P}$, define $w_p(x) = \sum_{\lambda_p \in D_p} \left| \left\langle f, T_{\lambda_p x^{-1}}\, g_p \right\rangle \right|^2$, so that $w(x) = \sum_{p \in \mathcal{P}} w_p(x)$. By Lemma 5, we can write w_p in the form

$$w_p(x) = \sum_{\gamma_p \in D_p^\perp} \hat{w}_p(\gamma_p)\, \gamma_p(x),$$

where

$$\hat{w}_p(\gamma_p) = s(D_p)^{-1} \int_{K \cap \gamma_p^{-1} K} \hat{f}(\omega)\, \overline{\hat{f}(\omega\gamma_p)}\, \overline{\hat{g}(\omega)}\, \hat{g}(\omega\gamma_p)\, d\omega,$$

and $K = \operatorname{supp} \hat{f}$ is a compact set (since $f \in \mathcal{D}$). We claim that $\{\hat{w}_p(\gamma_p) : p \in \mathcal{P}, \gamma_p \in D_p^\perp\}$ is in $l^1(\mathcal{P} \times D_p^\perp)$. To show that this is the case, we first apply Cauchy-Schwarz's inequality to the last expression and obtain:

$$\begin{aligned} s(D_p)\, \hat{w}_p(\gamma_p) &\leq \left(\int_{\gamma_p^{-1} K} |\hat{f}(\omega)\, \hat{g}(\omega\gamma_p)|^2\, d\omega \right)^{\frac{1}{2}} \left(\int_K |\hat{f}(\omega\gamma_p)\, \hat{g}(\omega)|^2\, d\omega \right)^{\frac{1}{2}} \\ &= \left(\int_K |\hat{f}(\omega\gamma_p^{-1})\, \hat{g}(\omega)|^2\, d\omega \right)^{\frac{1}{2}} \left(\int_K |\hat{f}(\omega\gamma_p)\, \hat{g}(\omega)|^2\, d\omega \right)^{\frac{1}{2}} . \end{aligned}$$

Next, using the inequality $2|cd| \leq |c|^2 + |d|^2$ together with (8) (since f satisfies the LIC), we have

$$\sum_{p \in \mathcal{P}} \sum_{\gamma_p \in D_p^\perp} |\hat{w}_p(\gamma_p)| < \infty,$$

which proves the claim. This in turn implies that

$$w(x) = \sum_{p\in\mathcal{P}} w_p(x) = \sum_{p\in\mathcal{P}} \sum_{\gamma_p\in D_p^\perp} \hat{w}_p(\gamma_p)\,\gamma(x),$$

where the convergence in the last sum is absolute and uniform. Finally, using the notation introduced at the beginning of this section, we can rewrite the last equality in the form:

$$w(x) = \sum_{\alpha\in\Lambda} \hat{w}(\alpha)\,\alpha(x),$$

where $\hat{w}(\alpha)$ is given by (11). Observe that $\{\hat{w}_p(\gamma_p) : p \in \mathcal{P}, \gamma_p \in D_p^\perp\} \in l^1(\mathcal{P} \times D_p^\perp)$ implies $\{\hat{w}(\alpha) : \alpha \in \Lambda\} \in l^1(\Lambda)$, and, thus, the Fourier series for w converges absolutely. $\square$

We can now prove Theorem 3

PROOF of Theorem 3. It suffices to prove the result for a dense subset of $L^2(G)$. The general case follows by a standard density argument.

We first prove that (ii) implies (i). By Proposition 6,

$$w(x) = \sum_{p\in\mathcal{P}} \sum_{\lambda_p\in D_p} \left|\left\langle T_x\, f, T_{\lambda_p}\, g_p\right\rangle\right|^2 = \sum_{\alpha\in\Lambda} \hat{w}(\alpha)\,\alpha(x),$$

where the last series converges absolutely (thus, $w(x)$ is continuous). Applying condition (ii) to $\hat{w}(\alpha)$, given by (11), we obtain

$$\hat{w}(\alpha) = \left(\int_{\widehat{G}} \hat{f}(\omega)\overline{\hat{f}(\omega\alpha)}\,d\omega\right)\delta_{\alpha,1}$$

for each $f \in \mathcal{D}$, where $\delta_{\alpha,1}$ is the Kronecker delta. Then (i) follows by setting $x = e$ in the expression for $w(x)$.

To prove the converse implication, let us assume that $N^2(f) = \|f\|^2$ for all $f \in L^2(G)$. Consider the function $z(x) = w(x) - \|f\|^2$. By Proposition 6, if $f \in \mathcal{D}$ then the function z is continuous and equals an absolutely convergent (generalized) trigonometric series whose coefficients are

$$\hat{z}(1) = \hat{w}(1) - \|f\|^2, \quad \text{and } \hat{z}(\alpha) = \hat{w}(\alpha),\ \alpha \neq 1.$$

By hypothesis, $z(x)=0$. Hence, applying [6, Theorem 7.12] (note that z is an almost periodic function) yields $\hat{z}(\alpha)=0$ for all $\alpha \in \Lambda$. Thus, for all $\alpha \in \Lambda$ and $f \in \mathcal{D}$, using (11) for the coefficients $\hat{w}(\alpha)$, we have

$$\int\limits_{\widehat{G}} \hat{f}(\omega)\, \overline{\hat{f}(\omega\alpha)} \left(\sum_{p \in \mathcal{P}_\alpha} s(D_p)^{-1}\, \overline{\hat{g}_p(\omega)}\, \hat{g}_p(\omega\alpha) \right) d\omega = \delta_{\alpha,1}\, \|f\|^2. \tag{12}$$

Observe that, by the LIC, the function h_α defined by

$$h_\alpha(\omega) = \sum_{p \in \mathcal{P}_\alpha} s(D_p)^{-1} \overline{\hat{g}_p(\omega)}\, \hat{g}_p(\omega\alpha),$$

with $\alpha \in \Lambda$ is locally integrable. In order to establish (ii), we need to show that $h_\alpha(\omega) = \delta_{\alpha,1}$, for a.e. $\omega \in \widehat{G}$. Consider first the case $\alpha = 1$. Arguing by contradiction, assume that $h_1(\omega) > 1$ for $\omega \in E$, where $\mu(E) > 0$. Let $\hat{f} = \chi_E$. Then

$$\int\limits_{\widehat{G}} |\hat{f}(\omega)|^2\, h_1(\omega)\, d\omega = \int\limits_{E} h_1(\omega)\, d\omega > \|f\|^2,$$

and this contradicts (12). A similar argument shows that one cannot have $h_1(\omega) < 1$ on any measurable set E of positive measure and, thus, $h_1(\omega) = 1$ for a.e. $\omega \in \widehat{G}$. Consider now the case $\alpha \neq 1$. Again, arguing by contradiction, assume that $h_\alpha(\omega) > 0$ for $\omega \in E$, where $\mu(E) > 0$. We can choose E small enough so that $E \bigcap E\alpha^{-1} = \emptyset$, for $\alpha \neq 1$. Let $\hat{f} = \chi_E + \chi_{(E\alpha^{-1})}$. Then

$$\int\limits_{\widehat{G}} \hat{f}(\omega)\, \overline{\hat{f}(\omega\alpha)}\, h_\alpha(\omega)\, d\omega = \int\limits_{E} h_\alpha(\omega)\, d\omega > 0,$$

and this contradicts (12). Thus $h_\alpha(\omega) = 0$, for a.e. $\omega \in \widehat{G}$, and this completes the proof. □

We can prove the following necessary condition for a family $\Phi_{\{D_p\}}^{\{g_p\}}$, given by (5), to form a Bessel system.

Proposition 7 *Let $\mathcal{P}$ be a countable set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(G)$, and $\{D_p : p \in \mathcal{P}\}$ a collection of uniform lattices in G. If the system $\Phi_{\{D_p\}}^{\{g_p\}}$, given by (5), is Bessel with constant B, then*

$$\sum_{p \in \mathcal{P}} s(D_p)^{-1}\, |\hat{g}_p(\omega)|^2 \le B \quad \text{for a.a. } \omega \in \widehat{G}. \tag{13}$$

PROOF. In all applications of this proposition that we will consider in this paper, $\mathcal{P}$ will be a subset of $\mathbb{Z}^r$ for some $r \in \mathbb{N}$. For simplicity we assume this to be the case here. However, the reader can easily check that this is not a loss of generality.

Since $\{T_{\lambda_p} g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$ is a Bessel sequence with constant B, then, for every $M \in \mathbb{N}$

$$\sum_{p\in\mathcal{P},|p|\le M} \sum_{\lambda_p \in D_p} |\left\langle f, T_{\lambda_p} g_p \right\rangle|^2 \le B\|f\|_2^2 \tag{14}$$

for all $f \in L^2(G)$. Applying Lemma 5 to the left hand side of this inequality to each $p \in \mathcal{P}$ (letting $x = e$), we have:

$$\sum_{p\in\mathcal{P},|p|\le M} \sum_{\gamma_p \in D_p^\perp} s(D_p)^{-1} \int_{\widehat{G}} \hat{f}(\omega)\, \overline{\hat{f}(\omega\,\gamma_p)}\, \overline{\hat{g}_p(\omega)}\, \hat{g}_p(\omega\,\gamma_p)\, d\omega \le B\|f\|_2^2 \tag{15}$$

for all $f \in \mathcal{D}$, $M \in \mathbb{N}$. Arguing by contradiction, let $\mathcal{B}(\omega_0, \delta)$ be a ball of radius $\delta > 0$ and center $\omega_0 \in \widehat{G}$, with respect to the metric d on $\widehat{G}$, and assume that

$$\sum_{p\in\mathcal{P},|p|\le M} s(D_p)^{-1}\, |\hat{g}_p(\omega)|^2 > B, \tag{16}$$

for a.e. $\omega \in \mathcal{B}(\omega_0, \delta)$, where δ is some positive constant. Next define f_ϵ by

$$\hat{f}_\epsilon(\omega) = \chi_{\mathcal{B}(\omega_0,\epsilon)}(\omega),$$

where $\epsilon < \min\{\delta, \delta_M/2\}$, and $\delta_M = \inf\{d(\gamma_p, 1) \in D_p^\perp \setminus \{1\} : |p| \le M\}$. Observe that $\delta_M > 0$ since there are only a finite number of elements $p \in \mathcal{P}$. Since $\epsilon < \delta_M/2$, then $\hat{f}_\epsilon(\omega)$ and $\hat{f}_\epsilon(\omega\,\gamma_p)$ have disjoint support for $\gamma_p \neq 1$. Using this observation, inequality (16) and the fact that $f_\epsilon \in \mathcal{D}$, we have that

$$\begin{aligned} &\sum_{p\in\mathcal{P},|p|\le M} \sum_{\gamma_p \in D_p^\perp} s(D_p)^{-1} \int_{\widehat{G}} \hat{f}_\epsilon(\omega)\, \overline{\hat{f}_\epsilon(\omega\,\gamma_p)}\, \overline{\hat{g}_p(\omega)}\, \hat{g}_p(\omega\,\gamma_p)\, d\omega \\ &\quad = \int_{\mathcal{B}(\omega_0,\epsilon)} |\hat{f}_\epsilon(\omega)|^2 \sum_{p\in\mathcal{P},|p|\le M} s(D_p)^{-1} |\hat{g}_p(\omega)|^2\, d\omega > B\, \|f_\epsilon\|^2, \end{aligned}$$

and this contradicts (15). The proof is completed by taking the limit for M tending to infinity in (14). □

4 Applications of the general characterization theorem

In this section, we study several applications of Theorem 3. We begin by considering the situation of locally compact abelian groups with compact connected component, and show that in this situation the LIC is equivalent to a simpler condition or can even be removed. Next we apply Theorem 3 to obtain a new characterization of Parseval frame generators for Gabor and affine sytems.

Throughout this subsection, let $\mathcal{P}$ be a countable index set, $\{g_p : p \in \mathcal{P}\}$ a family of functions in $L^2(G)$, and $\{D_p : p \in \mathcal{P}\}$ a collection of uniform lattices in G. As before, we define $\Lambda = \bigcup_{p\in\mathcal{P}} D_p^\perp$. Furthermore, let $\mathcal{D}$ be given by (7).

We start with the following simple observation.

Lemma 8 *If $\mathcal{P}$ is finite, then the system $\{T_{\lambda_p} g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$ satisfies the LIC.*

PROOF. Let K be a compact subset of $\widehat{G}$. Hence, for each $p \in \mathcal{P}$, there only exist finitely many $\gamma_p \in D_p^\perp$ with $K \cap \gamma_p^{-1} K \neq \emptyset$. Since $\mathcal{P}$ is supposed to be finite, both sums in the LIC, given by (6), are finite and hence there exists some $M < \infty$ such that

$$\sum_{p\in\mathcal{P}} s(D_p)^{-1} \sum_{\gamma_p \in D_p^\perp} \int_{K\cap\gamma_p^{-1}K} |\hat{g}_p(\omega)|^2 d\omega \leq M \sum_{p\in\mathcal{P}} s(D_p)^{-1} \|\hat{g}_p\|_2^2 < \infty. \qquad \square$$

4.1 LCA groups with compact connected component

We obtain the following general characterization of the LIC for LCA groups with compact connected component.

Proposition 9 *Let G be a LCA group with compact connected component and let H be some open compact subgroup of G. Then the following conditions are equivalent.*

(i) *The system $\{T_{\lambda_p} g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$ satisfies the LIC.*
(ii) *We have*

$$\sum_{p\in\mathcal{P}} s_H(D_p \cap H)^{-1} \int_K |\hat{g}_p(\omega)|^2 d\omega < \infty \quad \text{for all } K \subset \widehat{G} \text{ compact,}$$

where s_H denotes the lattice size with respect to H and the Haar measure m_H on H induced by m_G.

PROOF. Let D be some uniform lattice in G. We claim that

$$s(D)^{-1} \left(\#(D^\perp \cap H^\perp)\right) = s_H(D \cap H)^{-1}. \tag{17}$$

To prove this claim, we first choose a special fundamental domain S_H for D in G with respect to H. Since $D \cap H$ is a finite subgroup of H, there exists a fundamental domain $\tilde{S}_H$ for $D \cap H$ in H. Moreover, we have $[G : HD] < \infty$, where $[G : HD]$ is the index of HD in G, that is, the cardinality of G/HD. Thus we can choose a finite representative system $\{y_i : 1 \le i \le [G : HD]\}$ for the HD-cosets in G. Then we define the fundamental domain S_H by

$$S_H = \bigcup_{i=1}^{[G:HD]} y_i \tilde{S}_H.$$

Notice that this union is disjoint. It is straightforward to show that S_H is indeed a fundamental domain for D in G. Since the lattice size does not depend upon the fundamental domain we choose, we obtain

$$s(D) = m_G(S_H) = \sum_{i=1}^{[G:HD]} m_G(\tilde{S}_H) = [G : HD] s_H(D \cap H). \tag{18}$$

Secondly, we have $D^\perp \cap H^\perp = (HD)^\perp$, hence

$$\#(D^\perp \cap H^\perp) = \#((HD)^\perp) = [G : HD], \tag{19}$$

where the second equality follows from the fact that the dual group of a finite group is the group itself. Obviously, equations (18) and (19) yield our claim in (17).

Now we suppose that (i) holds. To show that this implies (ii), let K be a compact subset of $\widehat{G}$. Hence there exist $\tau_1, \dots, \tau_n \in \widehat{G}$ and corresponding $C_i \subseteq H^\perp$, $i = 1, \dots, n$, such that $K = \bigcup_{i=1}^n \tau_i C_i$. Thus it suffices to prove that

$$\sum_{p\in\mathcal{P}} s_H(D_p \cap H)^{-1} \int_{\bigcup_{i=1}^n \tau_i H^\perp} |\hat{g}_p(\omega)|^2 d\omega < \infty. \tag{20}$$

By [15, Remark 23.24(d)], $H^\perp$ is also compact, hence $\bigcup_{i=1}^n \tau_i H^\perp$ is compact. Since the LIC is satisfied, using (17) we have that

$$\begin{aligned}
\infty &> \sum_{p\in\mathcal{P}} s(D_p)^{-1} \sum_{\gamma_p \in D_p^\perp} \left(\int_{\bigcup_{i=1}^n \tau_i H^\perp \cap \bigcup_{j=1}^n \gamma_p^{-1} \tau_j H^\perp} |\hat{g}_p(\omega)|^2 d\omega \right) \\
&\ge \sum_{p\in\mathcal{P}} s(D_p)^{-1} \sum_{\gamma_p \in D_p^\perp \cap H^\perp} \left(\int_{\bigcup_{i=1}^n \tau_i H^\perp} |\hat{g}_p(\omega)|^2 d\omega \right)
\end{aligned}$$

$$= \sum_{p\in\mathcal{P}} s(D_p)^{-1}\,\#(D_p^\perp \cap H^\perp) \int\limits_{\bigcup_{i=1}^n \tau_i H^\perp} |\hat{g}_p(\omega)|^2 d\omega$$
$$= \sum_{p\in\mathcal{P}} s_H(D\cap H)^{-1} \int\limits_{\bigcup_{i=1}^n \tau_i H^\perp} |\hat{g}_p(\omega)|^2 d\omega.$$

This shows (i) $\Rightarrow$ (ii).

To prove the converse implication, suppose that (ii) holds. Let K be a compact subset of $\widehat{G}$. As above there exist $\tau_1,\ldots,\tau_n \in \widehat{G}$ and $C_i \subset H^\perp$, $i = 1,\ldots,n$, such that $K = \bigcup_{i=1}^n \tau_i C_i$. Since $\bigcup_{i=1}^n \tau_i H^\perp$ is a compact set and (ii) holds, we have

$$\infty > \sum_{p\in\mathcal{P}} s_H(D\cap H)^{-1} \int\limits_{\bigcup_{i=1}^n \tau_i H^\perp} |\hat{g}_p(\omega)|^2 d\omega$$
$$= \sum_{p\in\mathcal{P}} s(D_p)^{-1}\#(D_p^\perp \cap H^\perp) \int\limits_{\bigcup_{i=1}^n \tau_i H^\perp} |\hat{g}_p(\omega)|^2 d\omega.$$

Now we claim that

$$\#(\mathcal{D}_p^\perp \cap \bigcup_{i,j=1}^n \tau_i^{-1}\tau_j H^\perp) \le n^2 \cdot \#(\mathcal{D}_p^\perp \cap H^\perp). \tag{21}$$

Once this is shown we can continue the above computation to obtain

$$\infty > \sum_{p\in\mathcal{P}} s(D_p)^{-1}\#(D_p^\perp \cap H^\perp) \int\limits_{\bigcup_{i=1}^n \tau_i H^\perp} |\hat{g}_p(\omega)|^2 d\omega$$
$$\ge n^{-2} \sum_{p\in\mathcal{P}} s(D_p)^{-1} \sum_{\gamma_p\in D_p^\perp} \Bigg(\int\limits_{\bigcup_{i=1}^n \tau_i H^\perp \cap \bigcup_{j=1}^n \gamma_p^{-1}\tau_j H^\perp} |\hat{g}_p(\omega)|^2 d\omega \Bigg),$$

since $\bigcup_{i=1}^n \tau_i H^\perp \cap \bigcup_{j=1}^n \gamma_p^{-1}\tau_j H^\perp \neq \emptyset$ if and only if there exist $i,j \in \{1,\ldots,n\}$ with $\gamma_p \tau_j^{-1}\tau_i \in H^\perp$. This proves that the LIC holds for K.

Hence it remains to prove (21). Notice that it suffices to substitute $\bigcup_{i,j=1}^n \tau_i^{-1}\tau_j H^\perp$ by some set $\bigcup_{i=1}^{n^2} \alpha_i H^\perp$ with disjoint union. Now we choose the coset representatives of $H^\perp$ in $\widehat{G}$ to be $D \cup R$, where $D \subset \mathcal{D}_p^\perp$ and $\mathcal{D}_p^\perp \cap \alpha H^\perp = \emptyset$ for all $\alpha \in R$. Then, if $\alpha \in R$, we have $\#(\mathcal{D}_p^\perp \cap \alpha H^\perp) = 0$, and if $\alpha \in D$, we have

$$\#(\mathcal{D}_p^\perp \cap \alpha H^\perp) = \#(\alpha^{-1}\mathcal{D}_p^\perp \cap H^\perp) = \#(\mathcal{D}_p^\perp \cap H^\perp).$$

This proves (21) and hence (ii) implies (i). □

In the special cases of compact and discrete abelian groups, the preceding proposition yields the following equivalent conditions for the LIC to hold.

Corollary 10 *Let G be a compact abelian group. Then the following conditions are equivalent.*

(i) *The system $\{T_{\lambda_p} g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$ satisfies the LIC.*
(ii) *For all $\omega \in \widehat{G}$, we have*

$$\sum_{p \in \mathcal{P}} s(D_p)^{-1} |\hat{g}_p(\omega)|^2 < \infty.$$

PROOF. Choosing $H = G$ and $K = \{\gamma\}$ for some $\gamma \in \widehat{G}$ in Proposition 9, we obtain

$$\sum_{p \in \mathcal{P}} s_H(D_p \cap H)^{-1} \int_K |\hat{g}_p(\omega)|^2 d\omega = \sum_{p \in \mathcal{P}} s(D_p)^{-1} |\hat{g}_p(\gamma)|^2.$$

Note that it suffices to consider only $K = \{\gamma\}$, since each compact subset of $\widehat{G}$ contains only finitely many elements. □

Corollary 11 *Let G be a discrete abelian group. Then the following conditions are equivalent.*

(i) *The system $\{T_{\lambda_p} g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$ satisfies the LIC.*
(ii) *We have*

$$\sum_{p \in \mathcal{P}} \int_{\widehat{G}} |\hat{g}_p(\omega)|^2 d\omega < \infty.$$

PROOF. Choosing $H = \{e\}$ and $K = \widehat{G}$ in Proposition 9 yields

$$\sum_{p \in \mathcal{P}} s_H(D_p \cap H)^{-1} \int_K |\hat{g}_p(\omega)|^2 d\omega = \sum_{p \in \mathcal{P}} \int_{\widehat{G}} |\hat{g}_p(\omega)|^2 d\omega.$$

Notice that as before it suffices to consider only $K = \widehat{G}$. □

A natural question is the following. Let $\{T_{\lambda_p} g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$ be a system, which satisfies Theorem 3 (i) or (ii). Does this imply that this system then satisfies the LIC automatically, i.e., can we omit the hypothesis that the LIC has to be satisfied?

It will turn out, that this is true in some cases, however it is not a necessary condition. The compact and discrete groups will turn out to be the extreme cases.

Lemma 12 *Let G be a compact abelian group and suppose that the system $\{T_{\lambda_p} g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$ satisfies Theorem 3 (i) or (ii). Then it also satisfies the LIC.*

PROOF. If $\{T_{\lambda_p} g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$ satisfies Theorem 3 (i), it is, in particular, a Bessel system with constant B. Hence Proposition 7 shows that

$$\sum_{p \in \mathcal{P}} s(D_p)^{-1} |\hat{g}_p(\omega)|^2 \leq B, \quad \text{for all } \omega \in \widehat{G}.$$

This inequality also follows by Theorem 3 (ii) choosing $\alpha = 1$. Now we can apply Corollary 10, which finishes the proof. □

Lemma 13 *Let G be a discrete abelian group and let $\{T_{\lambda_p} g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$ be a Parseval frame for $L^2(G)$. Then this system does not necessarily satisfy the LIC.*

PROOF. Let $G = \mathbb{Z}$ and, for all $m \in \mathbb{Z}$ and $k \in \mathbb{N}$, let $[m]_k$ denote the residue class of m modulo k, and let $\sqcup$ denote a disjoint union. It is easy to see that, for all $m \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have

$$[m]_k = [m]_{2k} \sqcup [m+k]_{2k} \quad \text{and} \quad [m]_k = [m]_{2k} \sqcup [m-k]_{2k}. \tag{22}$$

Our aim is to write $\mathbb{Z}$ as a disjoint union of infinitely many pairwise different residue classes. For this, we start observing that $\mathbb{Z} = [0]_2 \sqcup [1]_2$. Applying (22) we obtain $\mathbb{Z} = [0]_2 \sqcup [1]_4 \sqcup [-1]_4$. Now we use the other formula in (22), which yields $\mathbb{Z} = [0]_2 \sqcup [1]_4 \sqcup [-1]_8 \sqcup [3]_8$. Iterating this procedure, using a simple induction argument, we obtain

$$\mathbb{Z} = \bigsqcup_{j \in \mathbb{N}} [a_j]_{2^j}\,, \tag{23}$$

where $a_j = \sum_{k=0}^{j-2} (-1)^k 2^k$, $j \in \mathbb{N}$.

Now we define $\mathcal{P} = \mathbb{N}$, $g_p = \chi_{\{a_p\}}$, and $D_p = 2^p\mathbb{Z}$, $p \in \mathcal{P}$. An easy calculation shows that $\hat{g}_p = e^{-2\pi i a_p \cdot}$. By (23), the family $\{T_{\lambda_p} g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$ equals $\{\chi_{\{m\}} : m \in \mathbb{Z}\}$, and hence is an orthonormal basis for $L^2(\mathbb{Z})$.

But this system does not satisfy the LIC. To prove this, by Corollary 11, we only have to compute

$$\sum_{p\in\mathcal{P}} \int_{\widehat{G}} |\hat{g}_p(\omega)|^2 d\omega = \sum_{p\in\mathbb{N}} \int_{\mathbb{T}} 1 d\omega,$$

which is not finite. □

4.2 Gabor systems

Let M_ω be the modulation operator, defined by

$$M_\omega f(x) = \omega(x)\, f(x), \quad \text{for } x \in G,\, \omega \in \widehat{G},$$

and consider the **Gabor systems:**

$$\mathcal{G}(\Psi) = \{T_\lambda\, M_\gamma\, \Psi :\, \lambda \in D, \gamma \in K\},$$

where $D \subset G$ is a uniform lattice, K is a discrete subset of $\widehat{G}$ and $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(G)$.

It is clear that we can write the system $\mathcal{G}(\Psi)$ in the form (5) by letting $\mathcal{P} = \{p = (\gamma, \ell) : \gamma \in K, \ell = 1, \ldots, L\}$, $D_p = D$ for each $p \in \mathcal{P}$ and $g_p = g_{(\gamma,\ell)} = M_\gamma\, \psi^\ell$.

We will deduce a characterization of all the functions Ψ such that $\mathcal{G}(\Psi)$ is a Parseval frame of $L^2(G)$. We start with the following observation.

Lemma 14 *Let $\{g_p : p \in \mathcal{P}\}$ be a countable family of functions in $L^2(G)$. Assume that $D_p = D$ for each $p \in \mathcal{P}$. If there is a constant C such that $\sum_{p\in\mathcal{P}} |\hat{g}_p(\omega)|^2 < C$, for a.e. $\omega \in \widehat{G}$, then the system $\{T_\lambda\, g_p : \lambda \in D, p \in \mathcal{P}\}$ satisfies the LIC.*

PROOF. Since K is compact, there only exist finitely many $\gamma \in D^\perp$ with $K \cap \gamma^{-1}K \neq \emptyset$, say M of them. A direct computation on the left hand side of (6) shows that:

$$\begin{aligned}
&\sum_{p\in\mathcal{P}} s(D)^{-1} \sum_{\gamma\in D^\perp} \int_{K\cap\gamma^{-1}K} |\hat{g}_p(\omega)|^2 d\omega \\
&\leq s(D)^{-1} C \sum_{\gamma\in D^\perp} \int_{K\cap\gamma^{-1}K} d\omega \\
&\leq s(D)^{-1}\, C\, M\, \mu(K) < \infty. \qquad \square
\end{aligned}$$

We now obtain the following characterization result.

Theorem 15 *$\mathcal{G}(\Psi)$ is a Parseval frame for $L^2(G)$ if and only if, for each $\lambda \in D^\perp$, we have:*

$$\sum_{\gamma \in K} \sum_{\ell=1}^{L} \hat{\psi}^\ell(\omega\gamma^{-1}) \, \overline{\hat{\psi}^\ell(\omega\gamma^{-1}\lambda)} = s(D)\, \delta_{1,\lambda} \quad \textit{for a.e. } \omega \in \widehat{G}. \tag{24}$$

PROOF. As we described before, the Gabor system $\mathcal{G}(\Psi)$ can be represented in the form (5). Using the notation of Theorem 3 for the system $\mathcal{G}(\Psi)$, we have that $\mathcal{P}_\alpha = \mathcal{P}$, $\Lambda = D^\perp$, and $\hat{g}_p = \hat{g}_{(\gamma,\ell)} = T_\gamma \hat{\psi}^\ell$. Also observe that, if $\mathcal{G}(\Psi)$ is a Parseval frame, then, by Proposition 7, for any $f \in \mathcal{D}$ there is a constant $B < \infty$ such that

$$\sum_{\gamma \in K} \sum_{\ell=1}^{L} s(D)^{-1} \, |\hat{\psi}^\ell(\omega\gamma^{-1})|^2 < B, \text{ for a.e. } \omega \in \widehat{G}.$$

This inequality also holds if equation (9) is satisfied (use $\alpha = 1$). Thus, by Lemma 14, we do not need to assume the LIC in Theorem 3. Finally, equation (24) follows from (9) $\square$

This theorem generalizes similar results in $L^2(\mathbb{R}^n)$, that one can find, for example, in [17,19,22].

4.3 Affine systems

In the theory of affine systems on $\mathbb{R}^n$, the elements of the family $\mathcal{W}_A(\psi)$, given by (2), are obtained under the action of translations and dilations on $\mathbb{R}^n$. These operations can be defined on a LCA group G by identifying the translations with the group action, and the dilations with the action of a group automorphism A on G (see [7] for a similar approach).

Let d be a metric on $\widehat{G}$. Without loss of generality (see [15, Vol.I, Sec.8]), the metric can be chosen to be translation–invariant, that is,

$$d(\alpha\omega, \beta\omega) = d(\alpha, \beta) \quad \text{for all } \alpha, \beta, \omega \in \widehat{G}.$$

Let A be a group automorphism on $\widehat{G}$, which is expanding, in the sense that

$$d(A(\alpha), A(\beta)) \geq c\, d(\alpha, \beta) \quad c > 1, \text{ for all } \alpha \neq \beta \in \widehat{G}. \tag{25}$$

We use the notation $A^2(\omega) = A(A(\omega))$, $A^0(\omega) = \omega$. Observe that A^{-1} is a contraction, since it follows from (25) that

$$d(A^{-1}(\alpha), A^{-1}(\beta)) \le c^{-1}\, d(\alpha,\beta) \quad c > 1, \text{ for all } \alpha \ne \beta \in \widehat{G}.$$

Throughout this section we fix a uniform lattice D in G and normalize the Haar measure on $\widehat{G}$ so that $s(D^\perp) = 1$.

Now let us consider the families $\Phi^{\{g_p\}}_{\{D_p\}}$, given by (5), where

$$\begin{aligned} &\mathcal{P} = \{(j,\ell) : j \in \mathbb{Z}, \ell = 1,\dots,L\},\ D_p^\perp = D_{(j,\ell)}^\perp = D_j^\perp = A^j(D^\perp), \\ &\text{and } \hat{g}_p(\omega) = \hat{g}_{(j,\ell)}(\omega) = \nu_j^{-1/2}\, \hat{\psi}^\ell(A^{-j}(\omega)), \end{aligned} \tag{26}$$

and the constant ν_j is chosen as $\nu_j := s(A^j(D^\perp))$, $j \in \mathbb{Z}$. Any family of this form will be called an **affine system** on $L^2(G)$ with respect to the automorphism A on $\widehat{G}$. The connection with the usual affine systems on $L^2(\mathbb{R}^n)$, that are related to the theory of wavelets, will be clarified later.

By Weil's formula (3) and the particular normalization of the Haar measure on $\widehat{G}$, it follows that, for each $\ell = 1,\dots,L$,

$$\int_{\widehat{G}} \nu_j^{-1}\, |\hat{\psi}^\ell(A^{-j}(\omega))|^2\, d\omega = \int_{\widehat{G}} |\hat{\psi}^\ell(\omega)|^2\, d\omega,$$

and this shows that the automorphism A, with the appropriate normalization, acts in a way similar to the unitary dilations in the case of classical wavelets (we will show in Example 20 that the unitary dilations are an automorphism on R^n).

Under these assumptions, the LIC, given by (6), is

$$L = \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{L} \sum_{\gamma\in D} \Big(\int_{K\cap(A^{-j}(\gamma)K)} |\hat{\psi}^\ell(A^{-j}(\omega))|^2 d\omega \Big) < \infty \tag{27}$$

for each compact subset K of $\widehat{G}$. The following proposition shows that the LIC is satisfied if A is an expanding automorphism on $\widehat{G}$.

Proposition 16 *Let G be a LCA group, and A be an expanding automorphism on $\widehat{G}$. Let $\mathcal{P}$, g_p and D_p be given by (26). If there is a constant C such that*

$$\sum_{p\in\mathcal{P}} s(D_p^\perp)\, |\hat{g}_p(\omega)|^2 = \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{L} |\hat{\psi}^\ell(A^{-j}(\omega))|^2 < C, \quad \textit{for a.e. } \omega \in \widehat{G}, \tag{28}$$

then the system $\{T_{\lambda_p} g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$ *satisfies the LIC (27).*

Before proving Proposition 16, we need some construction.

Given any $r > 0$, we use the notation $B(r) = \{g \in G : d(g,e) < r\}$ and $\widetilde{B}(r) = \{g \in G : 1/r < d(g,e) < r\}$. We obtain the following two lemmas.

Lemma 17 *Let* G *be a LCA group, and* A *be an expanding automorphism on* G*. Then there is a number* $N = N(A,r) \geq 0$ *such that*

$$\#\{j \in \mathbb{Z} : A^j(g) \in \widetilde{B}(r)\} \leq N, \quad \textit{for all } g \in G.$$

PROOF. If $g = e$, then $A^j(e) = e \notin \widetilde{B}(r)$, for any $j \in \mathbb{Z}$, and, thus, we can choose $N = 1$.

If $g \neq e$, let $j_0 = j_0(g)$ be the smallest integer such that $d(A^{j_0}(g), e) > 1/r$. This is possible since A is expanding, and there is a $c > 1$ such that $d(A^{-j}(g), e) \leq c^{-j}\, d(g,e)$, for all $j > 0$. Thus, $A^j(g) \notin \widetilde{B}(r)$ for all $j < j_0$. Next, choose $N_0 > 0$ such that $c^{N_0} > r^2$. Then, if $j \geq N_0$,

$$d(A^{j+j_0}(g), e) \geq c^j\, d(A^{j_0}(g), e) \geq c^{N_0}\, 1/r > r.$$

Thus, if $j \geq N_0$, then $A^{j+j_0}(g) \notin \widetilde{B}(r)$. It follows that

$$\{j \in \mathbb{Z} : A^j(g) \in \widetilde{B}(r)\} \subset \{j_0, j_0+1, \ldots, j_0+N_0-1\}.$$

The proof is completed by taking $N = N_0$ (observe that N_0 does not depend on j_0, and, in particular, does not depend on $g \in G$). $\square$

Lemma 18 *Let* G *be a LCA group,* $r > 0$*, and* A *be an expanding automorphism on* G*. Then there is a constant* $K = K(D,r) \geq 0$ *such that*

$$\#\{g \in D \setminus \{e\} : A^j(g) \in B(r)\} \leq K\, s(A^j(D))^{-1} s(D)^j.$$

PROOF. Let $S = \inf\{d(g,e) : g \in (D \setminus \{e\}) \bigcap B(r)\} > 0$. Since A is expanding, for any $r > 0$, there is a positive integer j such that

$$d(A^j(g), e) \geq c^j\, d(g,e) > c^j\, S > r,$$

for all $g \in D \setminus \{e\}$. Let j_1 be the smallest positive integer for which this holds. Thus, for all $j \geq j_1$,

$$\#\{g \in D \setminus \{e\} : A^j(g) \in B(r)\} = 0.$$

Next consider the case $j < j_1$. Let $g \in D \setminus \{e\}$ with $A^j(g) \in B(r)$ and let $h \in F$, where F denotes a fundamental domain of D. We set $T := \sup\{d(h,e) : h \in$

$F\}$. By [18, Lemma 2], the fundamental domain can be chosen to be relatively compact, hence $T < \infty$. Since A is expanding and $j < j_1$,

$$\begin{aligned} d(A^{-j_1+j}(gh), e) &= d(A^{-j_1+j}(g)A^{-j_1+j}(h), e) \\ &\le d(A^{-j_1+j}(g), e) + d(A^{-j_1+j}(h), e) \\ &< c^{-j_1} r + d(h, e) \\ &\le c^{-j_1} r + T =: R. \end{aligned}$$

Thus,

$$\{g \in D \setminus \{e\} : A^j(g) \in B(r)\} \subset \{g \in D : gF \subset A^{j_1-j}(B(R))\} =: \mathcal{M}_R^j.$$

Since the sets gF, $g \in D$ are disjoint, we obtain

$$\begin{aligned} \#\{g \in D \setminus \{e\} : A^j(g) \in B(r)\} \le \#\mathcal{M}_R^j &= \frac{m_G(\bigcup_{g \in \mathcal{M}_R^j} gF)}{s(D)} \\ &\le \frac{m_G(A^{j_1-j}(B(R)))}{s(D)}. \end{aligned}$$

For each measurable $Q \subset G$, Weil's formula implies

$$m_G(A(Q)) = \frac{s(A(D))}{s(D)} m_G(Q). \tag{29}$$

Applying this equation to the automorphisms A^{j_1} and A^{-j} yields

$$\#\{g \in D \setminus \{e\} : A^j(g) \in B(r)\} \le \frac{s(A^{j_1}(D))s(A^{-j}(D))}{s(D)^{j_1-j-1}} m_G(B(R)).$$

It remains to prove that $s(A^{-j}(D)) = c\, s(A^j(D))^{-1}$, for some constant c. Let Q be a fundamental domain for $A^{-1}(D)$ in G. Using the definition of a fundamental domain, it follows that $A^2(Q)$ is a fundamental domain for $A(D)$ in G. Then, applying (29) twice,

$$s(A(D)) = m_G(A^2(Q)) = \frac{s(A(D))^2}{s(D)^2} m_G(Q) = \frac{s(A(D))^2}{s(D)^2} s(A^{-1}(D)).$$

This implies

$$s(A^{-1}(D)) = s(A(D))^{-1} s(D)^2.$$

Now our claim follows from

$$s(A^{-j}(D)) = s((A^j)^{-1}(D)) = s(A^j(D))^{-1}s(D)^2.$$

Setting $K := \frac{s(A^{j_1}(D))}{s(D)^{j_1-3}} m_G(B(R))$ proves the proposition. $\square$

We can now prove Proposition 16.

PROOF of Proposition 16.

We write $L = L_1 + L_2$, where L_1 is the sum in (27) corresponding to $\gamma = 1$ and L_2 is the sum corresponding to $\gamma \in D^\perp \setminus \{1\}$.

It is clear that

$$L_1 = \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{L} \int_K |\hat{\psi}^\ell(A^{-j}(\omega))|^2 d\omega < \int_K C\, d\omega \, < \, \infty.$$

Let us consider now L_2. Choose $r > 0$ such that $K \subset \widetilde{B}(r)$. Using the change of variables $\eta = A^{-j}(\omega)$ (observe that $d(A^j(\eta)) = \nu_j\, d(\eta)$) we have:

$$\begin{aligned} L_2 &\leq \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{L} \sum_{\gamma\in D^\perp\setminus\{1\}} \int_{A^j(\eta)A^j(\gamma)\in\widetilde{B}(r)} |\hat{\psi}^\ell(\eta)|^2\, d(A^j(\eta)) \\ &= \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{L} \sum_{\gamma\in D^\perp\setminus\{1\}} \int_{A^j(\eta)A^j(\gamma)\in\widetilde{B}(r)} s(A^j(D^\perp))\, |\hat{\psi}^\ell(\eta)|^2\, d\eta. \end{aligned}$$

Observe that (use the triangle inequality for the metric d):

$$\begin{aligned} &\{\gamma \in D^\perp \setminus \{1\} : A^j(\eta) \in B(r) \text{ and } A^j(\eta)\, A^j(\gamma) \in B(r)\} \\ &\quad \subseteq \{\gamma \in D^\perp : A^j(\gamma) \in B(2r)\}, \end{aligned}$$

and, by Lemma 18,

$$\#\{\gamma \in D^\perp : A^j(\gamma) \in B(r)\} \leq K(D, r)\, s(A^j(D^\perp))^{-1},$$

since the Haar measure on $\widehat{G}$ is normalized so that $s(D^\perp) = 1$. This implies that

$$L_2 \leq K(D, r) \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{L} \int_{A^j(\eta)\in\widetilde{B}(r)} |\hat{\psi}^\ell(\eta)|^2\, d\eta.$$

Finally, using Lemma 17,

$$L_2 \leq K(D,r)\, N(A,r) \sum_{\ell=1}^{L} \|\psi^\ell\|_2^2 < \infty.$$

Thus, $L_1 + L_2 < \infty$ and this completes the proof. □

Using Proposition 16, we can now state the following characterization result for the affine systems. Notice that we do not need to assume the LIC in this theorem.

Theorem 19 *Let G be a LCA group, and A be an expanding automorphism on $\widehat{G}$. Then the affine system $\{T_{\lambda_p} g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$, where $\mathcal{P}$, g_p and D_p are given by (26), is a Parseval frame for $L^2(G)$ if and only if, for each $\alpha \in \bigcup_{j\in\mathbb{Z}} A^j(D^\perp)$, we have*

$$\sum_{(j,\ell)\in\mathcal{P}_\alpha} \overline{\hat{\psi}^\ell(A^{-j}(\omega))}\, \hat{\psi}^\ell(A^{-j}(\omega)\,\alpha) = \delta_{\alpha,1} \quad \text{for a.e. } \omega \in \widehat{G}, \tag{30}$$

where $\mathcal{P}_\alpha = \{(j,\ell) \in \mathcal{P} : \alpha \in A^j(D^\perp)\}$.

PROOF. We apply Theorem 3, where $\mathcal{P}$, g_p and D_p are given by (26). Equation (30) follows directly from (9). Thus, we only have to show that the LIC is satisfied. In order to do that, observe that, if the affine system $\{T_{\lambda_p} g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$ is a Parseval frame, then, by Proposition 16, for any $f \in \mathcal{D}$ there is a constant $C < \infty$ such that

$$\sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{L} |\hat{\psi}^\ell(A^{-j}(\omega))|^2 < C, \text{ for a.e. } \omega \in \widehat{G}.$$

This inequality also holds if equation (30) is satisfied (use $\alpha = 1$). Thus, by Proposition 28, we do not need to assume the LIC in Theorem 3, and this completes the proof. □

In the following, we apply Theorem 19 to some special LCA groups. As a first application, we consider the case $G = \mathbb{R}^n$, and we show that the usual affine systems on $L^2(\mathbb{R}^n)$ are easily described within our framework.

Example 20 Let $G = \mathbb{R}^n$ and $D = \mathbb{Z}^n$. Then $\widehat{G} = \mathbb{R}^n$, with the usual Euclidean metric, and $D^\perp = \mathbb{Z}^n$. The matrix $A \in GL_n(\mathbb{R})$, where all eigenvalues λ of A satisfy $|\lambda| > 1$, is an expanding group automorphism on $\widehat{G}$. Under these assumptions, from definitions (26), for $p \in \mathcal{P} = \{(j,\ell) : j \in \mathbb{Z}, \ell = 1,\ldots,L\}$

we have that $\hat{g}_p(\omega) = |\det A|^{-j/2}\,\hat{\psi}^\ell(A^{-j}\omega)$ and $D_p^\perp = A^j\,D^\perp = A^j\,\mathbb{Z}^n$. It follows that $g_p(x) = |\det B|^{j/2}\,\psi^\ell(B^j x)$, where $B = A^t$. Thus, the system $\{T_{\lambda_p}\,g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$ is the usual affine system on $L^2(\mathbb{R}^n)$:

$$\{T_{B^{-j}k}\,g_{(j,\ell)}(x) = |\det B|^{j/2}\,\psi^\ell(B^j x - k) : \; j \in \mathbb{Z}, k \in \mathbb{Z}^n, \ell = 1, \ldots, L\}. \quad (31)$$

From Theorem 19, we have that the affine system (31) is a Parseval frame for $L^2(\mathbb{R}^n)$ if and only if, for all $\alpha \in \Lambda = \bigcup_{j\in\mathbb{Z}} A^j\,\mathbb{Z}^n$

$$\sum_{(j,\ell)\in\mathcal{P}_\alpha} \hat{\psi}^\ell(A^{-j}\xi)\,\overline{\hat{\psi}^\ell(A^{-j}(\xi+\alpha))} = \delta_{\alpha,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \quad (32)$$

where, for $\alpha \in \Lambda$, $\mathcal{P}_\alpha = \{(j,\ell) \in \mathcal{P} : \; \alpha \in A^j\,\mathbb{Z}^n\}$. This result recovers Theorem 5.9 in [13] and, as shown in that paper, this result generalizes and contains all classical characterization results about affine systems, including those in [3,5,11,23]. We refer to the same paper for more detail about the motivation and the history of these and similar characterization equations for the affine systems in $L^2(\mathbb{R}^n)$.

Example 21 A *radial function* on $\mathbb{R}^n$ is a function $f : \mathbb{R}^n \to \mathbb{C}$, which satisfies $f(\|x\|) = f(\|y\|)$ for all $x, y \in \mathbb{R}^n$ with $\|x\| = \|y\|$. Radial functions occur in a natural way both in mathematics and applications, including, for example, the study of radar signals. In order to introduce an affine system, that can be used to decompose and analyze these types of signals, we define a new function $\psi : \mathbb{R}^+ \to \mathbb{C}$ by $\psi(r) := f(\|x\|)$ for all $r \in \mathbb{R}^+$, where $x \in \mathbb{R}^n$ is chosen such that $\|x\| = r$. Notice that the radial function f is uniquely determined by ψ and $f(0)$. Now we can apply our general method to the group $G = \mathbb{R}^+$. This is a locally compact abelian group with dual group $\mathbb{R}$. The character of G associated with some $y \in \mathbb{R}$, is the function $x \mapsto e^{2\pi i y \ln x}$. As a uniform lattice in G we choose $D = \{2^n : n \in \mathbb{Z}\}$. A simple calculation shows that $D^\perp = \frac{1}{\ln 2}\mathbb{Z}$. Now let $\mathcal{P} = \mathbb{Z}$ and let $A : \mathbb{R} \to \mathbb{R}$ be the expansive automorphism defined by $A(y) = (\ln 2)y$. Then $\nu_p = s(A^p(D^\perp)) = (\ln 2)^{p-1}$ for all $p \in \mathbb{Z}$. Further we define the functions g_p by

$$\hat{g}_p(y) = (\ln 2)^{\frac{1-p}{2}}\,\hat{\psi}((\ln 2)^{-p} y).$$

Then it follows that $g_p(x) = (\ln 2)^{\frac{p+1}{2}}\psi(x^{(\ln 2)^p})$. Observing that

$$T_{2^n}\,g_p(x) = (\ln 2)^{\frac{1-p}{2}}\,\psi((2^{-n}x)^{(\ln 2)^p}) = (\ln 2)^{\frac{1-p}{2}}\,\psi(e^{-n(\ln 2)^{p+1}}x^{(\ln 2)^p}),$$

we obtain the affine system

$$\Phi(\psi) = \{(\ln 2)^{\frac{p+1}{2}}\psi(e^{-n(\ln 2)^{p+1}}\,x^{(\ln 2)^p}) : \; p, n \in \mathbb{Z}\}.$$

Thus we can apply Theorem 19, which shows that $\Phi(\psi)$ is a Parseval frame for $L^2(\mathbb{R}^+)$ if and only if, for each $p \in \mathbb{Z}$ and $a \in A^p(D^\perp) = (\ln 2)^{p-1}\mathbb{Z}$, $a \neq 0$, we have

$$(\ln 2)^{2(1-p)} \overline{\hat{\psi}((\ln 2)^{-p}y)}\, \hat{\psi}((\ln 2)^{-p}(y+a)) = 0 \quad \text{for a.e. } y \in \mathbb{R}$$

and

$$\sum_{p\in\mathbb{Z}} (\ln 2)^{2(1-p)}\, |\hat{\psi}((\ln 2)^{-p}y)|^2 = 1 \quad \text{for a.e. } y \in \mathbb{R},$$

since $\mathcal{P}_a = \{q \in \mathbb{Z} : a \in (\ln 2)^{q-1}\mathbb{Z}\} = \{p\}$, if $a \in (\ln 2)^{p-1}\mathbb{Z}, a \neq 0$ and $\mathcal{P}_0 = \mathcal{P} = \mathbb{Z}$.

Example 22 Let us consider the subgroup of upper triangular matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y \in \mathbb{R}.$$

We can identify this group with $G = \mathbb{R}^2$ equipped with the group multiplication given by

$$(x_1, y_1)(x_2, y_2) = (x_1 + x_2, y_1 + y_2 - x_1x_2).$$

This is a locally compact abelian group with dual group $\mathbb{R}^2$. The character of G associated with some $z \in \mathbb{R}^2$, is the function $(x, y) \mapsto e^{2\pi i\left\langle z,(x,y+\frac{1}{2}x^2)\right\rangle}$. As a uniform lattice in G we choose $D = \mathbb{Z}^2$. A simple calculation shows that $D^\perp = \mathbb{Z} \times 2\mathbb{Z}$. Now let $\mathcal{P} = \mathbb{Z}$ and let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the expansive automorphism defined by $A(x, y) = B(x, y)^t$, where

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $\nu_p = s(A^p(D^\perp)) = 2^{2p+1}$ for all $p \in \mathbb{Z}$. Further we define g_p by

$$\hat{g}_p(x, y) = 2^{-p-\frac{1}{2}}\hat{\psi}(B^{-p}(x, y)^t).$$

Then we have that $g_p(x, y) = 2^{p-\frac{1}{2}}\, \psi(2^p x, 2^{p-1}(2y+(1-2^p)x^2))$. By the definition of the group multiplication we have $(m, n)^{-1} = (-m, -n-m^2)$. Observing that $T_{(m,n)}\, g_p(x,y) = \psi_{p,m,n}(x,y)$, where

$$\psi_{p,m,n}(x,y) = 2^{p-\frac{1}{2}}\psi(2^p(x-m), 2^{p-1}(2(y-n-m^2)+(1-2^p)(x-m)^2)),$$

we obtain the affine system

$$\Phi(\psi) = \{\psi_{p,m,n}(x,y) : p,m,n \in \mathbb{Z}\}.$$

Thus, we can apply Theorem 19, which shows that $\Phi(\psi)$ is a Parseval frame for $L^2(G)$ if and only if, for each $a \in \bigcup_{p\in\mathcal{P}} A^p(D^\perp) = \bigcup_{p\in\mathbb{Z}} B^p(\mathbb{Z}\times 2\mathbb{Z})$, we have

$$\sum_{p\in\mathcal{P}_a} 2^{-4p-2}\, \overline{\hat{\psi}(B^{-p}(x,y)^t)}\hat{\psi}(B^{-p}((x,y)+a)^t) = \delta_{a,0} \quad \text{for a.e. } (x,y)\in\mathbb{R}^2,$$

with $\mathcal{P}_a = \{p \in \mathbb{Z} : a \in B^p(\mathbb{Z}\times 2\mathbb{Z})\}$. Observe that one can deduce several variants of this construction for more general matrices $B \in GL_2(\mathbb{R})$.

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