Houston Journal of Mathematics
© University of Houston
Volume , No. ,

DENSITY OF FRAMES AND SCHAUDER BASES OF WINDOWED EXPONENTIALS

CHRISTOPHER HEIL AND GITTA KUTYNIOK

Communicated by Vern I. Paulsen

Abstract. This paper proves that every frame of windowed exponentials satisfies a Strong Homogeneous Approximation Property with respect to its canonical dual frame, and a Weak Homogeneous Approximation Property with respect to an arbitrary dual frame. As a consequence, a simple proof of the Nyquist density phenomenon satisfied by frames of windowed exponentials with one or finitely many generators is obtained. The more delicate cases of Schauder bases and exact systems of windowed exponentials are also studied. New results on the relationship between density and frame bounds for frames of windowed exponentials are obtained. In particular, it is shown that a tight frame of windowed exponentials must have uniform Beurling density.

1. Introduction

A sequence $\mathcal{F} = \{f_i\}_{i \in I}$ in a Hilbert space H is a *frame* for H if there exist $A, B > 0$ such that $A\,\|f\|^2 \le \sum_{i \in I} |\langle f, f_i\rangle|^2 \le B\,\|f\|^2$ for all $f \in H$. Frames provide unconditional basis-like (but generally non-unique) representations of vectors in H. They have applications in a wide range of areas, such as sampling theory [1], operator theory [16], nonlinear sparse approximation [14], wavelet theory [7], wireless communications [29], data transmission with erasures [13], signal processing [3], and quantum computing [8].

Received by the editors May 26, 2006.
2000 *Mathematics Subject Classification.* Primary 42C40; Secondary 46C99.
Key words and phrases. Density, frames, Gabor systems, Riesz bases, Schauder bases, windowed exponentials.
Research supported by NSF Grant DMS-0139261 and DFG Research Fellowship KU 1446/5.

Let Ω be a bounded subset of $\mathbf{R}^d$, let $g \in L^2(\Omega)$, and let Λ be a sequence of points in $\mathbf{R}^d$. Then

$$\mathcal{E}(g, \Lambda) \;=\; \{e^{2\pi i \lambda \cdot x} g(x)\}_{\lambda \in \Lambda}$$

is a system of *windowed exponentials* in $L^2(\Omega)$. These systems play important roles in the theory of reconstruction from irregular samples, e.g., [15], [1]. Necessary density conditions for $\mathcal{E}(g, \Lambda)$ to be a frame or Riesz basis for $L^2(\Omega)$ are known. In particular:

(a) if $\mathcal{E}(g, \Lambda)$ is a frame for $L^2(\Omega)$ then $D^-(\Lambda) \geq |\Omega|$, and
(b) if $\mathcal{E}(g, \Lambda)$ is a Riesz basis for $L^2(\Omega)$ then $D^-(\Lambda) = D^+(\Lambda) = |\Omega|$,

where $D^-(\Lambda)$, $D^+(\Lambda)$ denote the lower and upper Beurling densities of Λ, respectively (see Definition 2.1). The value $|\Omega|$ is the Nyquist density. The precise formulation of the Nyquist density is due to Landau [23], [24], in the context of sampling and interpolation of band-limited functions. There is a rich literature on this subject, as well as for related ideas concerning sampling in the Bargmann–Fock space of entire functions and necessary density conditions for Gabor frames in $L^2(\mathbf{R}^d)$. Extensions to systems $\bigcup_{k=1}^N \mathcal{E}(g_k, \Lambda_k)$ with finitely many generators are known and will be the setting of this paper, but for simplicity we will in this introduction discuss only the case of a single generator.

Proofs of (a) and (b) were given (with some restrictions) by Gröchenig and Razafinjatovo in [15], utilizing the idea of the Homogeneous Approximation Property that had been introduced for Gabor systems in [28] (cf. the extensions and applications in [6]). For a frame of windowed exponentials, the Homogeneous Approximation Property roughly states that for each $f \in L^2(\Omega)$ and $\varepsilon > 0$, there is a single $R > 0$ such that *every* modulation $M_\alpha f(x) = e^{2\pi i \alpha x} f(x)$ can be equally well-approximated (to within ε) using only those frame elements whose indices λ lie in the box $Q_R(\alpha)$ of sidelengths R centered at α. This is remarkable since Λ is not assumed to have any structure whatsoever—there need not be any relation between the points in $\Lambda \cap Q_R(\alpha)$ for different α.

We have the following main purposes in this paper. (1) We prove that every frame of windowed exponentials satisfies a *Strong Homogeneous Approximation Property* with respect to its canonical dual frame and a *Weak Homogeneous Approximation Property* with respect to an *arbitrary* dual frame. (2) We give a simple proof (without restrictions) of the Nyquist density phenomenon satisfied by frames of windowed exponentials. (3) We present new results and conjectures regarding Nyquist phenomena and the Homogeneous Approximation Property for the more delicate cases of Schauder bases and exact systems of windowed exponentials. (4) We present new results on the relationship between density and frame bounds for frames of windowed exponentials. In particular, we obtain the new result that every *tight* frame of windowed exponentials ($A = B$) must have *uniform* Beurling density. Tight frames are especially useful since the complexity of implementing frame algorithms is strongly tied to the ratio B/A.

Our paper is organized as follows. In Section 2 we present our notation and background on density, frames, and Schauder bases, and present some results that we will need regarding the properties of the Fourier transforms of compactly supported functions. Our main results are presented in Section 3. In Section 3.1 we show that Schauder bases and Bessel sequences must have finite upper density. We show in Section 3.2 that all frames of windowed exponentials satisfy a strong version of the HAP, and at least some Schauder bases of windowed exponentials satisfy a weak version of the HAP. Section 3.3 presents the Comparison Theorem, which shows that any frame or exact sequence which possesses this weak version of the HAP must have a density greater than any Schauder basic sequence of windowed exponentials. In Section 3.4 we derive necessary density conditions for frames of windowed exponentials and obtain relations among the density of the index set, the frame bounds, and the norms of the generators. Finally, in Section 3.5 we derive necessary density conditions for certain classes of Schauder bases and exact systems of windowed exponentials, and present some conjectures and open problems regarding these systems.

2. Notation and Preliminaries

In this section we define our terminology and provide some background, discussion, and examples related to our results.

2.1. **General notation.** Throughout, Ω will be a bounded, measurable subset of $\mathbf{R}^d$. We regard any function g with domain Ω as being defined on $\mathbf{R}^d$ by setting $g(x) = 0$ for $x \notin \Omega$. Consequently, $L^p(\Omega) \subset L^1(\mathbf{R}^d)$ for all $1 \leq p \leq \infty$. The Fourier transform of $g \in L^1(\mathbf{R}^d)$ is $\hat{g}(\xi) = \int_{\mathbf{R}^d} g(x)\, e^{-2\pi i \xi \cdot x}\, dx$.

Let $\Lambda = \{\lambda_i\}_{i \in I}$ be a sequence of points in $\mathbf{R}^d$, with countable or uncountable index set I (more precisely, this means that Λ is the function $i \mapsto \lambda_i$). For simplicity of notation, we will write $\Lambda \subset \mathbf{R}^d$, but we always mean that Λ is a sequence (in the above sense) and not merely a subset of $\mathbf{R}^d$. In particular, repetitions of elements in the sequence are allowed. We will say that Λ is a *lattice* if $\Lambda = A(\mathbf{Z}^d)$ where A is a $d \times d$ invertible matrix.

Often we will deal with several sequences $\Lambda_1, \dots, \Lambda_N \subset \mathbf{R}^d$, and will use the notation $\Lambda = \bigcup_{k=1}^N \Lambda_k$ to denote the disjoint union of these sequences. In particular, if each Λ_k is countable and is indexed as $\Lambda_k = \{\lambda_{jk}\}_{j\in\mathbf{N}}$, then Λ is the sequence $\Lambda = \{\lambda_{11}, \dots, \lambda_{1N}, \lambda_{21}, \dots, \lambda_{2N}, \dots\}$.

The *translation* of a function g by $\alpha \in \mathbf{R}^d$ is $T_\alpha g(x) = g(x-\alpha)$, and the *modulation* of g by $\beta \in \mathbf{R}^d$ is $M_\beta g(x) = e^{2\pi i\beta\cdot x} g(x)$. Using this notation, $\mathcal{E}(g,\Lambda) = \{M_\lambda g\}_{\lambda\in\Lambda}$.

Given $x \in \mathbf{R}^d$ and $h > 0$, we let $Q_h(x)$ denote the half-open cube in $\mathbf{R}^d$ centered at x with side lengths h, specifically, $Q_h(x) = \prod_{j=1}^d [x_j - \frac{h}{2}, x_j + \frac{h}{2})$.

If H is a Hilbert space and $f_i \in H$ for $i \in I$, then $\operatorname{span}\{f_i\}_{i\in I}$ will denote the finite linear span of $\{f_i\}_{i\in I}$, and $\overline{\operatorname{span}}\{f_i\}_{i\in I}$ will denote the closure of this set in H. The distance from a vector $f \in H$ to a closed subspace $V \subset H$ is $\operatorname{dist}(f,V) = \inf\{\|f-v\| : v \in V\} = \|f - P_V f\|$, where P_V is the orthogonal projection onto V.

2.2. **Beurling Density.** Beurling density measures in some sense the average number of points contained in unit cubes. The precise definition is as follows.

Definition 2.1. Let $\Lambda = \{\lambda_i\}_{i\in I}$ be a sequence of points in $\mathbf{R}^d$. The *lower* and *upper Beurling densities* of Λ are, respectively,

$$D^-(\Lambda) \;=\; \liminf_{h\to\infty} \inf_{\beta\in\mathbf{R}^d} \frac{\#\big(\Lambda \cap Q_h(\beta)\big)}{h^d},$$

$$D^+(\Lambda) \;=\; \limsup_{h\to\infty} \sup_{\beta\in\mathbf{R}^d} \frac{\#\big(\Lambda \cap Q_h(\beta)\big)}{h^d}.$$

In general $\sum_{k=1}^N D^-(\Lambda_k) \le D^-(\bigcup_{k=1}^N \Lambda_k) \le D^+(\bigcup_{k=1}^N \Lambda_k) \le \sum_{k=1}^N D^+(\Lambda_k)$, but these inequalities may be strict, e.g., consider $\Lambda_1 = \{k \in \mathbf{Z} : k \ge 0\}$ and $\Lambda_2 = \{k \in \mathbf{Z} : k < 0\}$.

A sequence Λ satisfies $D^+(\Lambda) < \infty$ if and only if $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N$ where each Λ_k is δ_k*-uniformly separated* for some $\delta_k > 0$, i.e., $\delta_k = \inf_{\lambda\ne\mu\in\Lambda_k} |\lambda-\mu| > 0$ for each k [6, Lem. 2.3].

The following additional implications of finite density will be useful later. If $D^+(\Lambda) < \infty$ then there must exist a finite constant K_0 such that $\#(\Lambda \cap Q_1(x)) \le K_0$ for every $x \in \mathbf{R}^d$. If $Q_h(x)$ is a cube with side lengths $h > 1$, then it is possible to cover $Q_h(x)$ using no more than $(h+1)^d$ cubes of side lengths 1. Therefore, for $x \in \mathbf{R}^d$ and $h > 1$ we have

$$\#\big(\Lambda \cap Q_h(x)\big) \;\le\; K_0\,(h+1)^d \;\le\; 2^d K_0\, h^d \;=\; 2^d K_0\, |Q_h(x)|.$$

Similar observations apply to other types of sets besides cubes. For example, we will later consider "square annuli" of the form $Q_{h+r}(x) \setminus Q_{h-r}(x)$. With r fixed, reasoning similar to the above shows that there is a constant K_1 such that for all $x \in \mathbf{R}^d$ and all h large enough,

$$\#\big(\Lambda \cap Q_{h+r}(x) \setminus Q_{h-r}(x)\big) \;\le\; K_1\, |Q_{h+r}(x) \setminus Q_{h-r}(x)| \;=\; K_1\, \big((h+r)^d - (h-r)^d\big).$$

More generally, these estimates are particular consequences of Landau's proof that the definition of Beurling density is unchanged if instead of dilating cubes we dilate any compact set with unit measure whose boundary has measure zero [24], cf. [22].

2.3. **Bases and Frames.** We use standard notations for Schauder bases as found in the texts [25], [30], [33], and standard notations for frames and Riesz bases as found in [5], [7], [33].

Definition/Facts 2.2. Let $\{f_i\}_{i\in\mathbf{N}}$ be a sequence in a Hilbert space H.

We say $\{f_i\}_{i\in\mathbf{N}}$ is *complete* if its finite linear span is dense in H. It is *minimal* if there exists a sequence $\{\tilde{f}_i\}_{i\in\mathbf{N}}$ in H that is biorthogonal to $\{f_i\}_{i\in\mathbf{N}}$, i.e., $\langle f_i, \tilde{f}_j\rangle = \delta_{ij}$ for $i,\, j \in \mathbf{N}$. Equivalently, $\{f_i\}_{i\in\mathbf{N}}$ is minimal if $f_j \notin \overline{\mathrm{span}}\{f_i\}_{i\ne j}$ for each $j \in \mathbf{N}$. A sequence that is both minimal and complete is called *exact*. In this case the biorthogonal sequence is unique.

We say $\{f_i\}_{i\in\mathbf{N}}$ is a *Schauder basis* if for each $f \in H$ there exist unique scalars c_i such that $f = \sum_{i=1}^{\infty} c_i f_i$. Every Schauder basis is exact, and the biorthogonal sequence $\{\tilde{f}_i\}_{i\in\mathbf{N}}$ is also a basis, called the *dual Schauder basis*. Further, we have

$$\forall\, f \in H, \quad f \;=\; \sum_{i=1}^{\infty} \langle f, \tilde{f}_i\rangle\, f_i \;=\; \sum_{i=1}^{\infty} \langle f, f_i\rangle\, \tilde{f}_i, \tag{2.1}$$

with uniqueness of the scalars in these expansions. The associated *partial sum operators* are $S_N(f) = \sum_{i=1}^{N} \langle f, \tilde{f}_i\rangle\, f_i$ for $f \in H$. The *basis constant* is the finite number $K = \sup_N \|S_N\|$.

A Schauder basis $\{f_i\}_{i\in\mathbf{N}}$ is *bounded* if there exist $C_1,\, C_2 > 0$ such that $C_1 \le \|f_i\| \le C_2$ for every i. The dual basis $\{\tilde{f}_i\}_{i\in\mathbf{N}}$ of a bounded Schauder basis is a bounded Schauder basis.

A Schauder basis $\{f_i\}_{i\in\mathbf{N}}$ is *unconditional* if the series in (2.1) converge unconditionally for every f. Consequently any countable index set can be used to index an unconditional basis.

A *Riesz basis* is the image of an orthonormal basis under a continuously invertible mapping of H onto itself. Every Riesz basis is a bounded unconditional basis, and conversely.

We say $\{f_i\}_{i\in\mathbf{N}}$ is a *frame* for H if there exist constants A, $B > 0$, called *frame bounds*, such that

$$\forall\, f \in H, \quad A\,\|f\|^2 \;\le\; \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \;\le\; B\,\|f\|^2. \tag{2.2}$$

All Riesz bases are frames, but not conversely.

If $\{f_i\}_{i\in\mathbf{N}}$ satisfies at least the second inequality in (2.2) then we say that $\{f_i\}_{i\in\mathbf{N}}$ is a *Bessel sequence* or that it *possesses an upper frame bound*, and we call B a *Bessel bound*. Likewise if at least the first inequality in (2.2) is satisfied then we say that $\{f_i\}_{i\in\mathbf{N}}$ *possesses a lower frame bound*.

A sequence $\{f_i\}_{i\in\mathbf{N}}$ is a Bessel sequence if and only if the *analysis operator* $Cf = \{\langle f, f_i\rangle\}_{i\in\mathbf{N}}$ is a bounded mapping $C\colon H \to \ell^2$. In this case, the adjoint of C is the *synthesis operator* $C^*\colon \ell^2 \to H$ given by $C^*(\{c_i\}_{i\in\mathbf{N}}) = \sum c_i f_i$ (the series converges unconditionally in the norm of H). In particular, if B is a Bessel bound then

$$\forall\, \{c_i\}_{i\in\mathbf{N}} \in \ell^2, \quad \Big\|\sum_{i\in\mathbf{N}} c_i f_i\Big\|^2 \;\le\; B \sum_{i\in\mathbf{N}} |c_i|^2. \tag{2.3}$$

Hence $\|f_i\|^2 \le B$ for every i, so every Bessel sequence is bounded above in norm.

If $\{f_i\}_{i\in\mathbf{N}}$ is a frame then the *frame operator* $Sf = C^*Cf = \sum \langle f, f_i\rangle\, f_i$ is a bounded, positive definite, invertible map of H onto itself.

If $\{f_i\}_{i\in\mathbf{N}}$ is a frame and $\{\tilde{g}_i\}_{i\in\mathbf{N}}$ is a sequence in H such that $f = \sum \langle f, \tilde{g}_i\rangle\, f_i$ for every $f \in H$, then $\{\tilde{g}_i\}_{i\in\mathbf{N}}$ is called a *dual sequence* to $\{f_i\}_{i\in\mathbf{N}}$. By Lemma 2.3 below, if a dual sequence is a Bessel sequence then it is also a frame, and hence is called a *dual frame* for $\{f_i\}_{i\in\mathbf{N}}$. We show in Lemma 2.3 that in this case we also have $f = \sum \langle f, f_i\rangle\, \tilde{g}_i$ for all $f \in H$.

Every frame $\mathcal{F} = \{f_i\}_{i\in\mathbf{N}}$ has a *canonical dual frame* $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in\mathbf{N}}$ given by $\tilde{f}_i = S^{-1} f_i$ where S is the frame operator. In particular,

$$\forall\, f \in H, \quad f \;=\; \sum_{i=1}^{\infty} \langle f, \tilde{f}_i\rangle\, f_i \;=\; \sum_{i=1}^{\infty} \langle f, f_i\rangle\, \tilde{f}_i, \tag{2.4}$$

and furthermore the series in (2.4) converges unconditionally for every f (so any countable index set can be used to index a frame). Equation (2.4) is identical to (2.1), but for a frame the coefficients in (2.4) need not be unique. In fact, if $\mathcal{F} = \{f_i\}_{i\in\mathbf{N}}$ is a frame and $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in\mathbf{N}}$ is its canonical dual frame, then the following statements are equivalent:

(i) $\mathcal{F}$ is a Riesz basis,
(ii) $\mathcal{F}$ is a Schauder basis,

(iii) the coefficients in (2.4) are unique for each $f \in H$,
(iv) $\mathcal{F}$ and its canonical dual frame $\tilde{\mathcal{F}}$ are biorthogonal.

In case any one of these hold, the canonical dual frame $\tilde{\mathcal{F}}$ coincides with the dual basis of $\mathcal{F}$.

We say that $\{f_i\}_{i\in\mathbf{N}}$ is a *frame sequence*, a *Riesz sequence*, or a *Schauder basic sequence* if it is a frame, Riesz basis, or Schauder basis for its closed linear span in H, respectively.

The following two results will be useful later.

Lemma 2.3. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a frame in a Hilbert space H. If $\mathcal{G} = \{\tilde{g}_i\}_{i\in I}$ is a dual sequence that is a Bessel sequence, then $\mathcal{G}$ is a frame, and furthermore,*

$$\forall\, f \in H, \quad f \;=\; \sum_{i\in I} \langle f, \tilde{g}_i\rangle\, f_i \;=\; \sum_{i\in I} \langle f, f_i\rangle\, \tilde{g}_i. \tag{2.5}$$

Proof. Let A, B be frame bounds for $\mathcal{F}$. Given $f \in H$, the first equality in (2.5) is the definition of dual sequence. Consequently,

$$\begin{aligned}\|f\|^4 \;=\; \langle f, f\rangle^2 \;=\; \biggl(\sum_{i\in I}\langle f, \tilde{g}_i\rangle\, \langle f_i, f\rangle\biggr)^2 \;&\le\; \biggl(\sum_{i\in I} |\langle f, \tilde{g}_i\rangle|^2\biggr)\biggl(\sum_{i\in I} |\langle f_i, f\rangle|^2\biggr)\\ &\le\; \biggl(\sum_{i\in I} |\langle f, \tilde{g}_i\rangle|^2\biggr)\, B\, \|f\|^2.\end{aligned}$$

Rearranging, we see that this implies that $\mathcal{G}$ has a lower frame bound of $\frac{1}{B}$. Since we already know that it has an upper frame bound, we conclude that $\mathcal{G}$ is a frame.

To show the second equality in (2.5), fix any $f \in H$. Since $\mathcal{G}$ is a frame and $\{\langle f, f_i\rangle\}_{i\in I} \in \ell^2$, the series $g = \sum \langle f, f_i\rangle\, \tilde{g}_i$ converges unconditionally to some element of H. However, for any $h \in H$ we have

$$\langle h, f\rangle \;=\; \Bigl\langle \sum_{i\in I}\langle h, \tilde{g}_i\rangle\, f_i,\, f\Bigr\rangle \;=\; \sum_{i\in I}\langle h, \tilde{g}_i\rangle\, \langle f_i, f\rangle \;=\; \Bigl\langle h,\, \sum_{i\in I}\langle f, f_i\rangle\, \tilde{g}_i\Bigr\rangle \;=\; \langle h, g\rangle,$$

so $f = g$. □

Lemma 2.4. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be an exact sequence in a Hilbert space H, and let $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$ be its biorthogonal sequence.*

(a) *If $\mathcal{F}$ is Bessel with Bessel bound B and $\tilde{\mathcal{F}}$ is complete, then $\tilde{\mathcal{F}}$ possesses a lower frame bound of $\frac{1}{B}$. Furthermore, $\tilde{\mathcal{F}}$ is exact and is norm-bounded below.*

(b) *If $\mathcal{F}$ possesses a lower frame bound A, then $\tilde{\mathcal{F}}$ is Bessel with Bessel bound $\frac{1}{A}$. Furthermore, $\tilde{\mathcal{F}}$ is norm-bounded above.*

PROOF. (a) We must show that $\frac{1}{B}\,\|f\|^2 \le \sum |\langle f, \tilde{f}_i\rangle|^2$ for all $f \in H$. If $\sum |\langle f, \tilde{f}_i\rangle|^2$ is infinite then this is trivial, so assume $\sum |\langle f, \tilde{f}_i\rangle|^2 < \infty$. Since $\mathcal{F}$ is Bessel, the series $g = \sum \langle f, \tilde{f}_i\rangle\, f_i$ converges unconditionally in H. By biorthogonality, we have $\langle g, \tilde{f}_i\rangle = \langle f, \tilde{f}_i\rangle$ for all $i \in I$, and since $\tilde{\mathcal{F}}$ is complete, this implies $g = f$. Consequently, by Cauchy–Schwarz and the fact that $\mathcal{F}$ is Bessel, we have

$$\begin{aligned} \|f\|^4 \;=\; \left(\sum_{i\in I} \langle f, \tilde{f}_i\rangle\, \langle f_i, f\rangle\right)^2 \;&\le\; \left(\sum_{i\in I} |\langle f, \tilde{f}_i\rangle|^2\right)\left(\sum_{i\in I} |\langle f, f_i\rangle|^2\right) \\ &\le\; \left(\sum_{i\in I} |\langle f, \tilde{f}_i\rangle|^2\right) B\, \|f\|^2. \end{aligned}$$

Rearranging therefore gives the desired inequality. Finally, by biorthogonality and the fact that $\mathcal{F}$ is Bessel, we have for each $j \in I$ that $1 = \sum_i |\langle \tilde{f}_j, f_i\rangle|^2 \le B\, \|\tilde{f}_j\|^2$, so $\tilde{\mathcal{F}}$ is norm-bounded below.

(b) Let $c = \{c_i\}_{i\in I} \in \ell^2(I)$ be given. If F is a finite subset of I, then by definition of lower frame bound and by biorthogonality,

$$A \left\| \sum_{i\in F} c_i \tilde{f}_i \right\|^2 \;\le\; \sum_{j\in I} \left| \left\langle \sum_{i\in F} c_i \tilde{f}_i,\, f_j \right\rangle \right|^2 \;=\; \sum_{i\in F} |c_i|^2. \tag{2.6}$$

Consequently, $Dc = \sum_{i\in I} c_i \tilde{f}_i$ converges unconditionally for each $c \in \ell^2(I)$, and the estimate in (2.6) extends to hold with I in place of F. Hence the synthesis operator $D\colon \ell^2(I) \to H$ is bounded with $\|D\|^2 \le \frac{1}{A}$. Therefore the analysis operator $C = D^*\colon H \to \ell^2(I)$ also satisfies $\|C\|^2 \le \frac{1}{A}$. Since $Cf = \{\langle f, \tilde{f}_i\rangle\}_{i\in I}$, we see that $\tilde{\mathcal{F}}$ is Bessel with Bessel bound $\frac{1}{A}$, and hence is bounded above in norm. □

2.4. **Examples in $L^2[-\frac{1}{2}, \frac{1}{2}]$.** We give now some basic results and illustrative examples of sequences of windowed exponentials on the domain $\Omega = [-\frac{1}{2}, \frac{1}{2}]$.

Lemma 2.5. *Given $g \in L^2[-\frac{1}{2}, \frac{1}{2}]$ and $\Lambda = \mathbf{Z}$, the following statements hold.*

(a) *$\mathcal{E}(g, \mathbf{Z})$ is a Bessel sequence in $L^2[-\frac{1}{2}, \frac{1}{2}]$ with Bessel bound $B > 0$ if and only if $g \in L^\infty[-\frac{1}{2}, \frac{1}{2}]$ with $|g(x)|^2 \le B$ a.e. In this case the optimal Bessel bound is $B = \|g\|_\infty^2$.*

(b) *$\mathcal{E}(g, \mathbf{Z})$ is a frame for $L^2[-\frac{1}{2}, \frac{1}{2}]$ with frame bounds A, $B > 0$ if and only if $A \le |g(x)|^2 \ge B$ a.e.*

(c) *$\mathcal{E}(g, \mathbf{Z})$ is complete in $L^2[-\frac{1}{2}, \frac{1}{2}]$ if and only if $g(x) \ne 0$ a.e.*

(d) *$\mathcal{E}(g, \mathbf{Z})$ is minimal in $L^2[-\frac{1}{2}, \frac{1}{2}]$ if and only if $1/g \in L^2[-\frac{1}{2}, \frac{1}{2}]$. In this case, $\mathcal{E}(g, \mathbf{Z})$ is exact.*

(e) $\mathcal{E}(g, \mathbf{Z})$ *is a frame with frame bounds* A, $B > 0$ *if and only if* $A \le |g(x)|^2 \le B$ *a.e. In this case the frame is a Riesz basis.*

PROOF. (a) Assume that $\mathcal{E}(g, \mathbf{Z})$ is Bessel. Then for $f \in L^\infty[-\frac{1}{2}, \frac{1}{2}]$ we have that

$$\|f\bar{g}\|_2^2 \;=\; \sum_{n\in\mathbf{Z}} |(f\bar{g})^\wedge(n)|^2 \;=\; \sum_{n\in\mathbf{Z}} |\langle f, M_n g\rangle|^2 \;\le\; B\,\|f\|_2^2,$$

the first equality following from the fact that $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$ forms an orthonormal basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$. Rearranging, we see that $\int_{-1/2}^{1/2} |f(x)|^2\,(B - |g(x)|^2)\,dx \ge 0$. If we had $|g(x)|^2 > B$ on some set S of positive measure, then taking $f = \chi_S$ would yield a contradiction. Hence $|g(x)|^2 \le B$ a.e. The converse is similar.

(b), (c), (e) These proofs are similar to the proof of statement (a).

(d) If $1/g \in L^2[-\frac{1}{2}, \frac{1}{2}]$, then $\tilde{\mathcal{E}} = \mathcal{E}(1/\bar{g}, \mathbf{Z}) = \{e^{2\pi inx}/\overline{g(x)}\}_{n\in\mathbf{Z}}$ is biorthogonal to $\mathcal{E}(g, \mathbf{Z})$.

Conversely, suppose that there exists some biorthogonal system $\tilde{\mathcal{E}} = \{\tilde{g}_n\}_{n\in\mathbf{Z}}$ to $\mathcal{E}(g, \mathbf{Z})$. Then for each $m \in \mathbf{Z}$ we have

$$\delta_{mn} \;=\; \langle \tilde{g}_m, M_n g\rangle \;=\; (\tilde{g}_m \bar{g})^\wedge(n).$$

Since $\tilde{g}_m \bar{g} \in L^1[-\frac{1}{2}, \frac{1}{2}]$ and Fourier coefficients of L^1 functions are unique, we conclude that $\tilde{g}_m(x)\overline{g(x)} = e^{2\pi imx}$ a.e. Consequently, we have that $1/g(x) = \overline{\tilde{g}_0(x)} \in L^2[-\frac{1}{2}, \frac{1}{2}]$. □

It is interesting to note that there is no known example of a set of non-windowed exponentials $\mathcal{E}(\chi_{[-\frac{1}{2},\frac{1}{2}]}, \Lambda) = \{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda}$ which forms a Schauder basis but not a Riesz basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$, and no proof that such a system cannot exist. However, we see in the next example that there do exist windowed exponentials which form a Schauder basis but not a Riesz basis. In this example we have $\Lambda = \mathbf{Z}$, so $D^\pm(\Lambda) = 1 = |\Omega|$.

Example 2.6. Fix $0 < \alpha < \frac{1}{2}$. Then $g(x) = |x|^\alpha$ and $\tilde{g}(x) = |x|^{-\alpha}$ both belong to $L^2[-\frac{1}{2}, \frac{1}{2}]$. Therefore $\mathcal{E}(g, \mathbf{Z}) = \{e^{2\pi inx}|x|^\alpha\}_{n\in\mathbf{Z}}$ and $\mathcal{E}(\tilde{g}, \mathbf{Z}) = \{e^{2\pi inx}|x|^{-\alpha}\}_{n\in\mathbf{Z}}$ are biorthogonal systems in $L^2[-\frac{1}{2}, \frac{1}{2}]$ and hence are minimal. It is a deeper result, due to Babenko [2], that these systems are actually Schauder bases for $L^2[-\frac{1}{2}, \frac{1}{2}]$ (see also the discussion in [30, pp. 351–354]). Since these systems are obtained by taking the orthonormal basis $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$ and performing an operation that is not a continuous bijection (i.e., multiplying by the function $|x|^\alpha$ which has a zero or by the unbounded function $|x|^{-\alpha}$), they are not Riesz bases. On the other hand, these systems do possess one but not both frame bounds. Specifically,

$\{e^{2\pi inx}|x|^{\alpha}\}_{n\in\mathbf{Z}}$ is a Bessel sequence while $\{e^{2\pi inx}|x|^{-\alpha}\}_{n\in\mathbf{Z}}$ possesses a lower frame bound.

The following is an example of an exact system of windowed exponentials in $L^2[-\frac{1}{2},\frac{1}{2}]$ which is not a Schauder basis. In this example we have $\Lambda = \mathbf{Z}\backslash\{0\}$, so $D^{\pm}(\Lambda) = 1 = |\Omega|$.

Example 2.7. Set $g(x) = x$, and note that $1/g(x) = 1/x \notin L^2[-\frac{1}{2},\frac{1}{2}]$. We will show that $\mathcal{E}(g,\mathbf{Z}\backslash\{0\}) = \{xe^{2\pi inx}\}_{n\neq 0}$ is an exact system in $L^2[-\frac{1}{2},\frac{1}{2}]$ that is not a Schauder basis. Since $\Lambda \neq \mathbf{Z}$ in this example, we cannot appeal to Lemma 2.5 to show these facts.

To show completeness, suppose $f \in L^2[-\frac{1}{2},\frac{1}{2}]$ is such that $\langle f(x), xe^{2\pi inx}\rangle = 0$ for all $n \neq 0$. Then the function $xf(x) \in L^1[-\frac{1}{2},\frac{1}{2}]$ satisfies $\langle xf(x), e^{2\pi inx}\rangle = 0$ for $n \neq 0$. Since Fourier coefficients of L^1-functions are unique, we conclude that $xf(x) = c$ a.e., where c is a constant. If $c \neq 0$ then we have $f(x) = c/x \notin L^2[-\frac{1}{2},\frac{1}{2}]$, which is a contradiction. Therefore $c = 0$, and hence $f = 0$ a.e. Thus $\mathcal{E}(g,\mathbf{Z}\backslash\{0\})$ is complete.

Now define $\tilde{g}_n(x) = \frac{e^{2\pi inx}-1}{x}$ for $n \neq 0$. Note that

$$\|\tilde{g}_n\|_2^2 \;=\; 8\,(-1)^n - 8 + 8\pi n \int_0^{\frac{1}{2}} \frac{\sin 2\pi nx}{x}\,dx \;<\; \infty,$$

so $\tilde{g}_n \in L^2[-\frac{1}{2},\frac{1}{2}]$. Further, for m, $n \neq 0$,

$$\langle xe^{2\pi imx}, \tilde{g}_n(x)\rangle \;=\; \langle e^{2\pi imx}, e^{2\pi inx} - 1\rangle \;=\; \delta_{mn}.$$

Therefore $\tilde{\mathcal{E}} = \{\tilde{g}_n\}_{n\neq 0}$ is biorthogonal to $\mathcal{E}(g,\mathbf{Z}\backslash\{0\})$, and thus each of these sequences are minimal in $L^2[-\frac{1}{2},\frac{1}{2}]$. In particular, $\mathcal{E}(g,\mathbf{Z}\backslash\{0\})$ is exact.

Now, if $\mathcal{E}(g,\mathbf{Z}\backslash\{0\})$ was a Schauder basis, then it would be a bounded basis, and hence its dual basis would also be a bounded basis. But since $\int_0^{\frac{1}{2}} \frac{\sin 2\pi nx}{x}\,dx \to \frac{\pi}{2}$ as $n \to \infty$, we see that $\|\tilde{g}_n\|_2$ is not uniformly bounded above in norm. Therefore, while $\mathcal{E}(g,\mathbf{Z}\backslash\{0\})$ is exact, it is not a Schauder basis.

Let us make some remarks concering the preceding example. Consider the following question. Let F be a finite subset of $\mathbf{Z}$. If we remove elements corresponding to F from the orthonormal basis $\{e^{2\pi inx}\}_{n\in\mathbf{Z}}$, will there always exist a function g such that $\mathcal{E}(g,\mathbf{Z}\backslash F) = \{e^{2\pi inx}\}_{n\in\mathbf{Z}}$ is complete? Example 2.7 shows that if $F = \{0\}$, then the answer is yes, with $g(x) = x$. In their classic paper on multiplicative completion, Boas and Pollard showed that the answer is yes for for finite subset F [4]. There are a surprising number of equivalent reformulations and interesting related results, for which we refer to the paper by Kazarian and Zink [20] and the references contained therein.

Expanding on the preceding paragraph, completeness is of course quite a different issue than being a basis. The fact remarked on in Example 2.7 that $\mathcal{E}(g, \mathbf{Z}\backslash\{0\})$ is not a Schauder basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$ is just a special case of the more general results proved by Kazarian in [18], [19]. In particular, it is shown there that if $F \subset \mathbf{Z}$ is finite and nonempty, then $\mathcal{E}(g, \mathbf{Z}\backslash F)$ can never be a Schauder basis for $L^p[-\frac{1}{2}, \frac{1}{2}]$ for any g.

The preceding remarks concern questions regarding systems $\mathcal{E}(g, \Lambda)$ that are derived in some way from an orthonormal basis. The main thrust of this paper is different; we are concerned with systems where Λ is not required to have any structure whatsoever. In particular, Λ need not be a subset of a lattice.

Following are a few further remarks concerning Example 2.7.

Remark 2.8. (a) Example 2.7 can also be interpreted in terms of sampling from derivatives. By taking the Fourier transform, the fact that $\mathcal{E}(g, \mathbf{Z}\backslash\{0\})$ is exact but not a Schauder basis implies that any function $f \in L^2(\mathbf{R})$ that is bandlimited to $[-\frac{1}{2}, \frac{1}{2}]$ (i.e., $\operatorname{supp}(\hat{f}) \subset [-\frac{1}{2}, \frac{1}{2}]$) is determined by the samples $\{f'(n)\}_{n\neq 0}$ of its derivative, but f cannot be stably reconstructed from these values in general.

(b) Example 2.7 behaves similarly to the case of the Gabor system $\mathcal{G}(\gamma, \mathbf{Z}^2) = \{e^{2\pi i m x}\gamma(x-n)\}_{m,n\in\mathbf{Z}}$ in $L^2(\mathbf{R})$ generated by the Gaussian function $\gamma(x) = e^{-x^2}$. By applying the Zak transform, it can be shown that $\mathcal{G}(\gamma, \mathbf{Z}^2)$ is overcomplete by a single element, meaning that one but not two elements can be removed and still leave a complete set. Furthermore, $\mathcal{G}(\gamma, \mathbf{Z}^2\backslash(0,0))$ is exact but is not a Schauder basis for $L^2(\mathbf{R})$.

(c) In Example 2.7, if we include the index $n = 0$ the system $\mathcal{E}(x, \mathbf{Z})$ is still not a frame for $L^2[-\frac{1}{2}, \frac{1}{2}]$, and similarly $\mathcal{E}(x - \frac{1}{2}, \mathbf{Z})$ is not a frame, but the two-generator system $\mathcal{E}(x, \mathbf{Z}) \cup \mathcal{E}(x - \frac{1}{2}, \mathbf{Z})$ is a frame for $L^2[-\frac{1}{2}, \frac{1}{2}]$, since

$$\sum_{n\in\mathbf{Z}} |\langle f(x), xe^{2\pi i n x}\rangle|^2 + \sum_{n\in\mathbf{Z}} |\langle f(x), (x - \tfrac{1}{2})e^{2\pi i n x}\rangle|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \bigl(x^2 + (x - \tfrac{1}{2})^2\bigr)\, |f(x)|^2\, dx,$$

and $x^2 + (x - \frac{1}{2})^2$ is bounded both above and below.

(d) In Example 2.6, the dual basis of the system of windowed exponentials $\mathcal{E}(g, \mathbf{Z})$ was another system of windowed exponentials $\mathcal{E}(\tilde{g}, \mathbf{Z})$. However, in Example 2.7, the system biorthogonal to $\mathcal{E}(g, \mathbf{Z}\backslash\{0\})$ was not itself a system of windowed exponentials. This is not a consequence of the fact that this second example is not a basis, but rather is due to the fact that the index set $\mathbf{Z}\backslash\{0\}$ is not a lattice.

In the following example, we see a system Λ which contains subsets that are lattices, but which is not itself a lattice. In particular, Λ is not uniformly separated.

Example 2.9. This example is a special case of the results in [32]. Let $\Lambda_1 = \frac{1}{2\sigma_1}\mathbf{Z}$, $\Lambda_2 = \frac{1}{2\sigma_2}\mathbf{Z} \setminus \{0\}$, and $\Lambda_3 = \{0\}$, where $\sigma_1 = \frac{1}{2(1+\sqrt{2})}$ and $\sigma_2 = \frac{\sqrt{2}}{2(1+\sqrt{2})}$. Note that the two lattices $\frac{1}{2\sigma_1}\mathbf{Z}$ and $\frac{1}{2\sigma_2}\mathbf{Z}$ intersect only at 0, so Λ_1 and Λ_2 are disjoint. Further, $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ (disjoint union, preserving the repetition of the point 0) is not uniformly separated, but does have uniform Beurling density $D^{\pm}(\Lambda) = 2\sigma_1 + 2\sigma_2 = 1 = |\Omega|$. It is shown in [32] that

$$\begin{aligned}\mathcal{E} &= \mathcal{E}(\chi_{[-\frac{1}{2},\frac{1}{2}]}, \Lambda_1) \cup \mathcal{E}(\chi_{[-\frac{1}{2},\frac{1}{2}]}, \Lambda_2) \cup \mathcal{E}(x\chi_{[-\frac{1}{2},\frac{1}{2}]}, \Lambda_3) \\ &= \{e^{\pi inx/\sigma_1}\}_{n\neq 0} \cup \{e^{\pi inx/\sigma_2}\}_{n\neq 0} \cup \{1, x\}\end{aligned}$$

has the following properties.

(a) $\mathcal{E}$ is exact, is Bessel but not a frame, and is not a Schauder basis for $L^2[-\frac{1}{2},\frac{1}{2}]$.
(b) The biorthogonal system of $\mathcal{E}$ is exact, is unbounded above in norm, and is not Bessel but does possess a lower frame bound.
(c) Although not a Schauder basis, $\mathcal{E}$ does satisfy the somewhat more general definition of a *basis with braces*, or a *Riesz basis from subspaces*.

2.5. **Existence of Frames and Riesz sequences in $L^2(\Omega)$.** The so-called *Spectral Set* or *Fuglede Conjecture* is that given $\Omega \subset \mathbf{R}^d$, there exists a sequence $\Lambda \subset \mathbf{R}^d$ such that $\mathcal{E}(\chi_\Omega, \Lambda)$ is an orthonormal basis for $L^2(\Omega)$ if and only if Ω tiles $\mathbf{R}^d$ (with overlaps of measure zero) under a set of translations. This conjecture has recently been shown to be false for $d \geq 4$ (but is open in lower dimensions), see [31], [21], [26].

In contrast, we show in the following lemma that, as long as the boundary of Ω has measure zero, it is always possible to construct a system of windowed exponentials with finitely many generators that forms an orthonormal sequence or a tight frame for $L^2(\Omega)$, and furthermore the density of the corresponding index set lies within ε of the Lebesgue measure of Ω.

Lemma 2.10. *Let Ω be a bounded subset of $\mathbf{R}^d$ such that $|\partial\Omega| = 0$, and let $\varepsilon > 0$ be given.*

(a) *There exist functions* $g_1, \dots, g_M \in L^2(\Omega)$ *and lattices* $\Delta_1, \dots, \Delta_M$ *such that* $\bigcup_{k=1}^M \mathcal{E}(g_k, \Delta_k)$ *is an orthonormal sequence in* $L^2(\Omega)$*, and furthermore* $|\Omega| - \varepsilon \le D^{\pm}(\Delta) \le |\Omega|$*, where* $\Delta = \bigcup_{k=1}^M \Delta_k$*.*

(b) *There exist functions* $h_1, \dots, h_N \in L^2(\Omega)$ *and lattices* $\Gamma_1, \dots, \Gamma_N$ *such that* $\bigcup_{k=1}^N \mathcal{E}(h_k, \Gamma_k)$ *is a Parseval frame for* $L^2(\Omega)$*, and furthermore,* $|\Omega| \le D^{\pm}(\Gamma) \le |\Omega| + \varepsilon$*, where* $\Gamma = \bigcup_{k=1}^N \Gamma_k$*.*

Proof. (a) Since $|\partial\Omega| = 0$, we may assume that Ω is open, and therefore there exist cubes $R_1, \dots, R_M \subset \Omega$ with disjoint interiors such that $\Omega_0 = \bigcup_{k=1}^M R_k$ satisfies $|\Omega| - \varepsilon \le |\Omega_0| \le |\Omega|$. Set $g_k = |R_k|^{-1/2} \chi_{R_k}$. Since R_k is a cube, there exists a lattice Δ_k contained in $\mathbf{R}^d$ such that $\mathcal{E}(g_k, \Delta_k)$ is an orthonormal basis for $L^2(R_k)$. Then $\bigcup_{k=1}^M \mathcal{E}(g_k, \Delta_k)$ is an orthonormal basis for $L^2(\Omega_0)$ and is an orthonormal sequence in $L^2(\Omega)$. Because the Δ_k are lattices we have $D^{\pm}(\Delta_k) = |R_k|$ and $D^{\pm}(\bigcup_{k=1}^M \Delta_k) = \sum_{k=1}^M D^{\pm}(\Delta_k) = \sum_{k=1}^M |R_k| = |\Omega_0|$.

(b) Since $|\partial\Omega| = 0$, we may assume that Ω is open, and therefore there exist cubes $S_1, \dots, S_N$ with disjoint interiors such that $\Omega_1 = \bigcup_{k=1}^N S_k$ contains Ω and satisfies $|\Omega| \le |\Omega_1| \le |\Omega| + \varepsilon$. Set $f_k = |S_k|^{-1/2} \chi_{S_k}$. Then there exist lattices Γ_k such that $\bigcup_{k=1}^N \mathcal{E}(f_k, \Gamma_k)$ is an orthonormal basis for $L^2(\Omega_1)$. The mapping $f \mapsto f\chi_\Omega$ is an orthogonal projection of $L^2(\Omega_1)$ onto $L^2(\Omega)$, and the image of an orthonormal basis under an orthogonal projection is a tight frame. Therefore, if we set $h_k = f_k \chi_\Omega$ then $\bigcup_{k=1}^N \mathcal{E}(h_k, \Gamma_k)$ is a tight frame for $L^2(\Omega)$, and by rescaling by an appropriate constant we can make it a Parseval frame. Further, $D^{\pm}(\bigcup_{k=1}^M \Gamma_k) = \sum_{k=1}^M D^{\pm}(\Gamma_k) = \sum_{k=1}^M |S_k| = |\Omega_1|$. □

2.6. **Amalgam Space Properties of the Fourier Transform.** Special cases of amalgam spaces were first introduced by Wiener, and subsequently many other special cases were introduced in the literature. A comprehensive general theory of amalgam spaces was introduced and extensively studied by Feichtinger, e.g., see [9], [10], [11]. For an expository introduction to Wiener amalgams with extensive references to the original literature, we refer to [17].

For our purposes, we will require only the following special case of Wiener amalgams.

Definition 2.11. Given $1 \le p, q \le \infty$, and $\alpha > 0$, the Wiener amalgam $W(L^p, \ell^q)$ consists of all functions F on $\mathbf{R}^d$ for which

$$\|F\|_{W(L^p,\ell^q)} \;=\; \left(\sum_{k \in \mathbf{Z}^d} \|F \cdot \chi_{Q_\alpha(\alpha k)}\|_p^q\right)^{1/q} \;<\; \infty,$$

with the usual adjustment if $q = \infty$.

$W(L^p, \ell^q)$ is a Banach space, and its definition is independent of the value of α in the sense that each choice of α yields the same vector space under an equivalent norm. The space

$$W(\mathcal{C}, \ell^q) \;=\; \{F \in W(L^\infty, \ell^q) : F \text{ is continuous}\}$$

is a closed subspace of $W(L^\infty, \ell^q)$. Note that we have the inclusion

$$L^p(\mathbf{R}^d) \;=\; W(L^p, \ell^p) \;\subset\; W(L^1, \ell^p),$$

and furthermore by [12, Thm. 7.1(b)] (compare [17, Thm. 11.8.3]) we have the following convolution relations:

$$\begin{aligned} L^p(\mathbf{R}^d) * W(L^\infty, \ell^1) \;&\subset\; W(L^1, \ell^p) * W(L^\infty, \ell^1) \\ &\subset\; W(L^1 * L^\infty, \ell^p * \ell^1) \;\subset\; W(L^\infty, \ell^p), \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} L^1(\mathbf{R}^d) * W(L^\infty, \ell^p) \;&=\; W(L^1, \ell^1) * W(L^\infty, \ell^p) \\ &\subset\; W(L^1 * L^\infty, \ell^1 * \ell^p) \;\subset\; W(L^\infty, \ell^p), \end{aligned} \tag{2.8}$$

with corresponding norm inequalities

$$\begin{aligned} \|F * G\|_{W(L^\infty, \ell^p)} \;&\le\; C\, \|F\|_p\, \|G\|_{W(L^\infty, \ell^1)}, \\ \|F * G\|_{W(L^\infty, \ell^p)} \;&\le\; C\, \|F\|_1\, \|G\|_{W(L^\infty, \ell^p)}, \end{aligned}$$

where C is a constant independent of F and G.

We now derive some amalgam space properties of the Fourier transforms of compactly supported functions.

Proposition 2.12. *Let Ω be a bounded subset of $\mathbf{R}^d$, and let $1 \le p \le 2$ be given. Then for $f \in L^2(\Omega)$ we have*

$$\hat{f} \in L^p(\mathbf{R}^d) \;\implies\; \hat{f} \in W(\mathcal{C}, \ell^p).$$

In particular, $\hat{f} \in W(\mathcal{C}, \ell^2)$ for every $f \in L^2(\Omega)$, and we have the norm estimate

$$\|\hat{f}\|_{W(\mathcal{C}, \ell^2)} \;=\; \|\hat{f}\|_{W(L^\infty, \ell^2)} \;\le\; C\, \|\hat{f}\|_2 \;=\; C\, \|f\|_2.$$

Proof. Let φ be any compactly supported function such that $\varphi(x) = 1$ for $x \in \Omega$ and $\hat{\varphi} \in W(\mathcal{C}, \ell^1)$. Suppose $f \in L^2(\Omega)$ is such that $\hat{f} \in L^p(\mathbf{R}^d)$. Then $f = f\varphi$, so by (2.7) we have $\hat{f} = \hat{f} * \hat{\varphi} \in L^p(\mathbf{R}^d) * W(L^\infty, \ell^1) \subset W(L^\infty, \ell^p)$, with a corresponding norm estimate. Since $f \in L^1(\Omega)$, we also have that $\hat{f}$ is continuous, and therefore $\hat{f} \in W(\mathcal{C}, \ell^p)$. □

As a corollary, we obtain a sufficient condition for $\mathcal{E}(g, \Lambda)$ to be a Bessel sequence.

Corollary 2.13. *Let $\Omega \subset \mathbf{R}^d$ be bounded. If $g \in L^2(\mathbf{R}^d)$ and $\Lambda \subset \mathbf{R}^d$ are such that*

(a) $\hat{g} \in L^1(\mathbf{R}^d)$*, and*
(b) $D^+(\Lambda) < \infty$*,*

then $\mathcal{E}(g, \Lambda)$ is a Bessel sequence in $L^2(\Omega)$.

Proof. Since $D^+(\Lambda) < \infty$, we can write $\Lambda = \bigcup_{k=1}^N \Lambda_k$ where each Λ_k is δ_k-uniformly separated for some $\delta_k > 0$. Let $\delta = \min\{\delta_1/2, \dots, \delta_N/2\}$. Then any cube with side lengths δ can contain at most one element of any Λ_k, and each element of Λ_k lies in some cube of the form $Q_\delta(n\delta)$ with $n \in \mathbf{Z}^d$. Given $f \in L^2(\Omega)$ we have $(f\bar{g})^\wedge = \hat{f} * \hat{\bar{g}} \in W(L^\infty, \ell^2) * L^1(\mathbf{R}^d) \subset W(L^\infty, \ell^2)$ by (2.8). Consequently,

$$\begin{aligned}
\sum_{\lambda \in \Lambda} |\langle f, M_\lambda g\rangle|^2 &= \sum_{k=1}^N \sum_{\lambda \in \Lambda_k} |(f\bar{g})^\wedge(\lambda)|^2 \\
&\le \sum_{k=1}^N \sum_{n \in \mathbf{Z}^d} \sup_{\xi \in Q_\delta(\delta n)} |(f\bar{g})^\wedge(\xi)|^2 \\
&= N \, \|(f\bar{g})^\wedge\|^2_{W(L^\infty, \ell^2)} \\
&= N \, \|\hat{f} * \hat{\bar{g}}\|^2_{W(L^\infty, \ell^2)} \\
&\le C_1 N \, \|\hat{f}\|^2_{W(L^\infty, \ell^2)} \, \|\hat{\bar{g}}\|_1^2 \;\le\; C_2 N \, \|\hat{g}\|_1^2 \, \|f\|_2^2,
\end{aligned}$$

so $\mathcal{E}(g, \Lambda)$ is Bessel with Bessel bound $C_2 N \, \|\hat{g}\|_1^2$. □

3. Density of Windowed Exponentials

We now develop our main results on the density of systems of windowed exponentials, and on the relationships between density and frame bounds for frames of windowed exponentials.

3.1. **Upper Density Estimates.**

Olson and Zalik proved in [27] that a necessary condition for a system $\{g(x - \alpha)\}_{\alpha \in \Gamma}$ of pure translates to be a Schauder basis for $L^p(\mathbf{R})$ is that Γ be uniformly separated. We prove an analogous result for Schauder basic sequences of windowed exponentials. For this result, no Hilbert space structure is needed; instead the essential ingredient is the continuity properties of modulation. For example, the spaces $L^p(\Omega)$ satisfy the hypotheses of the following lemma. The isometry hypothesis can be weakened further so that more function spaces are included; we omit the details. Although an exact sequence need not be uniformly separated (compare Example 2.9), we do not know if every minimal sequence of windowed exponentials must have finite density.

Lemma 3.1. *Let X be any Banach space of complex-valued functions defined on $\mathbf{R}^d$ such that:*

(a) *modulation is an isometry on X, i.e., $\|M_\beta f\| = \|f\|$ for all $f \in X$ and $\beta \in \mathbf{R}^d$, and*

(b) *modulation is strongly continuous, i.e., $\lim_{\beta\to 0} \|M_\beta f - f\| = 0$ for all $f \in X$.*

Let $g \in X$ and $\Lambda \subset \mathbf{R}^d$ be such that $\mathcal{E}(g,\Lambda)$ is a Schauder basic sequence in X. Then Λ is uniformly separated, i.e., $\delta = \inf_{\lambda\neq\mu\in\Lambda} |\lambda - \mu| > 0$. In particular, $D^+(\Lambda) < \infty$.

Proof. By hypothesis, $\mathcal{E}(g,\Lambda)$ is a Schauder basis for its closed span $Y = \overline{\text{span}}(\mathcal{E}(g,\Lambda))$ within X. Therefore there exists an ordering $\mathcal{E}(g,\Lambda) = \{M_{\lambda_n} g\}_{n\in\mathbf{N}}$ and a biorthogonal system $\tilde{\mathcal{E}} = \{\tilde{g}_n\}_{n\in\mathbf{N}} \subset Y^*$ such that

$$f \;=\; \sum_{k=1}^{\infty} \langle f, \tilde{g}_k\rangle\, M_{\lambda_k} g, \qquad f \in Y.$$

Let $S_N \colon Y \to Y$ be the partial sum operators $S_N(f) = \sum_{k=1}^{N} \langle f, \tilde{g}_k\rangle\, M_{\lambda_k} g$, $f \in Y$.

Fix $\varepsilon > 0$. Since modulation is strongly continuous, there exists a constant $h > 0$ such that $\|M_\beta g - g\| < \varepsilon$ whenever $|\beta| < h$. Suppose that Λ was not uniformly separated. Then there exist $m < n$ such that $|\lambda_m - \lambda_n| < h$, and, consequently, if we set

$$f_{m,n} \;=\; M_{\lambda_m} g - M_{\lambda_n} g,$$

then we have $\|f_{m,n}\| < \varepsilon$. Since $\mathcal{E}(g,\Lambda)$ and $\tilde{\mathcal{E}}$ are biorthogonal,

$$S_m(f_{m,n}) \;=\; \sum_{k=1}^{m} \langle M_{\lambda_m} g, \tilde{g}_k\rangle\, M_{\lambda_k} g \;-\; \sum_{k=1}^{m} \langle M_{\lambda_n} g, \tilde{g}_k\rangle\, M_{\lambda_k} g \;=\; M_{\lambda_m} g,$$

and hence $\|S_m(f_{m,n})\| = \|g\|$. But then

$$\|S_m\| \;=\; \sup_{\|f\|=1} \|S_m(f)\| \;\geq\; \frac{\|S_m(f_{m,n})\|}{\|f_{m,n}\|} \;>\; \frac{\|g\|}{\varepsilon}.$$

Since ε is arbitrary, this contradicts the fact that $\mathcal{E}(g,\Lambda)$ has a finite basis constant $K = \sup_N \|S_N\|$. □

We show next that any nontrivial Bessel sequence $\mathcal{E}(g,\Lambda)$ in $L^2(\Omega)$ must have finite upper density. While this does not imply that Λ is uniformly separated, it does imply that Λ must be the union of at most finitely many separated sequences.

Lemma 3.2. *Let Ω be a bounded subset of $\mathbf{R}^d$. If $g \in L^2(\Omega) \setminus \{0\}$ and $\Lambda \subset \mathbf{R}^d$ are such that $\mathcal{E}(g,\Lambda)$ is a Bessel sequence in $L^2(\Omega)$, then $D^+(\Lambda) < \infty$.*

Proof. Assume that $D^+(\Lambda) = \infty$. Choose any $f \in L^2(\Omega)$ with $\|f\|_2 = 1$. Then $f\bar{g} \in L^1(\Omega) \subset L^1(\mathbf{R}^d)$, so $(f\bar{g})^\wedge$ is continuous on $\mathbf{R}^d$. Since $(f\bar{g})^\wedge$ is not the zero function, it must be bounded away from zero on some cube $Q_\delta(\gamma)$, i.e., there exist some $\gamma \in \mathbf{R}^d$ and $\delta > 0$ such that

$$m \;=\; \inf_{\omega \in Q_\delta(\gamma)} |(f\bar{g})^\wedge(\omega)| \;>\; 0.$$

Now choose any $N > 0$. Since $D^+(\Lambda) = \infty$, there exists some cube $Q_\delta(\xi)$ which contains at least N elements of Λ. Set $h = M_{\xi-\gamma} f$. If $\lambda \in Q_\delta(\xi)$, then $\lambda - \xi + \gamma \in Q_\delta(\gamma)$, so

$$\begin{aligned}
\sum_{\lambda \in \Lambda} |\langle h, M_\lambda g\rangle|^2 \;&\geq\; \sum_{\lambda \in \Lambda \cap Q_\delta(\xi)} |\langle M_{\xi-\gamma} f, M_\lambda g\rangle|^2 \\
&=\; \sum_{\lambda \in \Lambda \cap Q_\delta(\xi)} |\langle f, M_{\lambda-\xi+\gamma} g\rangle|^2 \\
&=\; \sum_{\lambda \in \Lambda \cap Q_\delta(\xi)} |(f\bar{g})^\wedge(\lambda - \xi + \gamma)|^2 \;\geq\; N m^2.
\end{aligned}$$

Since $\|h\|_2 = \|f\|_2 = 1$ and N is arbitrary, it follows that $\mathcal{E}(g,\Lambda)$ cannot possess an upper frame bound. □

Lemmas 3.1 and 3.2 extend immediately to the case of finitely many generators. For example, if $\bigcup_{k=1}^N \mathcal{E}(g_k,\Lambda_k)$ is a Bessel sequence then each individual system $\mathcal{E}(g_k,\Lambda_k)$ is itself a Bessel sequence and therefore each Λ_k must have finite density by Lemma 3.2.

3.2. **Definition and Basic Properties of the HAP.** Ramanathan and Steger [28] proved (with some restrictions) that for Gabor frames in $L^2(\mathbf{R}^d)$, the frame expansions in (2.4) have a certain kind of uniformity of convergence with respect to translations and modulations. This property is called the Homogeneous Approximation Property (HAP), and it is a key property in deriving density results for Gabor frames.

A version of the HAP (with restrictions on the generators) for frames of translates of band-limited functions was proved by Gröchenig and Razafinjatovo in [15]. By applying the Fourier transform, this yields a HAP for frames of windowed exponentials. We will show in Section 3.3 that any frame of windowed exponentials, without restrictions, possesses a strong version of the HAP with respect to its canonical dual frame, and furthermore possesses a weak HAP with respect to *any* dual frame. We further show in Section 3.4 that a Schauder basis of windowed exponentials possesses a weak version of the HAP if the generating function satisfies some extra conditions, or if at least a lower frame condition is satisfied. Thus, we do not only recover the results of [15], but we extend the impact of the HAP to new settings. Further, in comparison to the results of Ramanathan and Steger (which applied to Gabor frames rather than systems of windowed exponentials), even considering only the case of the HAP for the canonical dual frame, our results apply far more generally—we do not need to assume that the index set has positive and finite density, we are not restricted to one dimension or to a single generator, and we avoid the considerable technicality of their approach (which relied on weak convergence of sequences).

Recall that, in general, if $\mathcal{E}(g,\Lambda)$ is a Schauder basis or a frame for $L^2(\Omega)$, then the dual basis or dual frame will be some set of functions $\tilde{\mathcal{E}} = \{\tilde{g}_\lambda\}_{\lambda\in\Lambda}$ from $L^2(\Omega)$, but it need not itself form a set of windowed exponentials. The same remarks apply to the case of systems with multiple generators.

Definition 3.3 (HAP)**.** Let Ω be a bounded subset of $\mathbf{R}^d$. Let $g_1,\dots,g_N \in L^2(\Omega)$ and $\Lambda_1,\dots,\Lambda_N \subset \mathbf{R}^d$ be such that $\mathcal{E} = \bigcup_{k=1}^N \mathcal{E}(g_k,\Lambda_k) = \bigcup_{k=1}^N \{M_\lambda g_k\}_{\lambda\in\Lambda_k}$ is a frame or an exact sequence for $L^2(\Omega)$. Let $\tilde{\mathcal{E}} = \bigcup_{k=1}^N \{\tilde{g}_{\lambda,k}\}_{\lambda\in\Lambda_k}$ be either any dual sequence (if $\mathcal{E}$ is a frame) or the biorthogonal system in $L^2(\Omega)$ (if $\mathcal{E}$ is exact). For $h>0$ and $\alpha\in\mathbf{R}^d$, set

$$W(h,\alpha) \;=\; \overline{\operatorname{span}}\big\{\tilde{g}_{\lambda,k} : \lambda\in\Lambda_k\cap Q_h(\alpha),\ k=1,\dots,N\big\}. \tag{3.1}$$

(a) We say that $\mathcal{E}$ possesses the *Weak Homogeneous Approximation Property (Weak HAP) with respect to* $\tilde{\mathcal{E}}$ if for each $f\in L^2(\Omega)$,

$$\begin{aligned} &\forall\,\varepsilon>0,\quad \exists\,R=R_W(f,\varepsilon)>0 \quad\text{such that}\quad \forall\,\alpha\in\mathbf{R}^d,\\ &\qquad \operatorname{dist}\big(M_\alpha f,\, W(R,\alpha)\big) < \varepsilon. \end{aligned} \tag{3.2}$$

(b) We say that $\mathcal{E}$ possesses the *Strong Homogeneous Approximation Property (Strong HAP) with respect to* $\tilde{\mathcal{E}}$ if for each $f\in L^2(\Omega)$,

$$\begin{aligned} &\forall\,\varepsilon>0,\quad \exists\,R=R_S(f,\varepsilon)>0 \quad\text{such that}\quad \forall\,\alpha\in\mathbf{R}^d,\\ &\qquad \Big\| M_\alpha f - \sum_{k=1}^N \sum_{\lambda\in\Lambda_k\cap Q_R(\alpha)} \langle M_\alpha f, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k}\Big\|_2 \;<\; \varepsilon. \end{aligned} \tag{3.3}$$

We refer to $R_W(f,\varepsilon)$ or $R_S(f,\varepsilon)$ as *associated radius functions.*

Remark 3.4. (a) We will simply write that "$\mathcal{E}$ satisfies the Weak HAP" if $\mathcal{E}$ satisfies the Weak HAP with respect to its canonical dual frame (if $\mathcal{E}$ is a frame) or its biorthogonal sequence (if $\mathcal{E}$ is exact), and similarly for the Strong HAP.

(b) Note that in Definition 3.3, if $\bigcup_{k=1}^N \mathcal{E}(g_k, \Lambda_k)$ is a frame or Schauder basis for $L^2(\Omega)$, then each subsequence $\mathcal{E}(g_k, \Lambda_k)$ is either a Bessel sequence or a Schauder basis for its closed span in $L^2(\Omega)$. Lemmas 3.1 or 3.2 therefore imply that each Λ_k has finite density. Consequently, each $\Lambda_k \cap Q_h(\alpha)$ is a finite set, and therefore each $W(h, \alpha)$ is finite-dimensional in this case.

Lemma 3.5. *Using the same notation as Definition 3.3, the following statements hold.*

(a) *The Strong HAP implies the Weak HAP.*
(b) *If $\mathcal{E}$ is a Riesz basis, then the Weak HAP implies the Strong HAP.*

PROOF. (a) The function $\sum_{k=1}^N \sum_{\lambda \in \Lambda_k \cap Q_R(\alpha)} \langle M_\alpha f, M_\lambda g_k \rangle \, \tilde{g}_{\lambda,k}$ is one element of the space $W(R, \alpha)$, so

$$\operatorname{dist}\bigl(M_\alpha f, \, W(R, \alpha)\bigr) \;\le\; \biggl\| M_\alpha f - \sum_{k=1}^N \sum_{\lambda \in \Lambda_k \cap Q_R(\alpha)} \langle M_\alpha f, M_\lambda g_k \rangle \, \tilde{g}_{\lambda,k} \biggr\|_2 .$$

Therefore the Strong HAP implies the Weak HAP.

(b) Assume that $\mathcal{E}$ is a Riesz basis that satisfies the Weak HAP. Then the biorthogonal system $\tilde{\mathcal{E}}$ is also a Riesz basis, and if A, B are frame bounds for $\mathcal{E}$, then $\frac{1}{B}$, $\frac{1}{A}$ are frame bounds for $\tilde{\mathcal{E}}$, so

$$\frac{1}{B} \sum_{k=1}^N \sum_{\lambda \in \Lambda_k} |a_{\lambda,k}|^2 \;\le\; \biggl\| \sum_{k=1}^N \sum_{\lambda \in \Lambda_k} a_{\lambda,k} \, \tilde{g}_{\lambda,k} \biggr\|_2^2 \;\le\; \frac{1}{A} \sum_{k=1}^N \sum_{\lambda \in \Lambda_k} |a_{\lambda,k}|^2$$

for any square-summable sequence of scalars $(a_{\lambda,k})$, cf. [33, p. 27]. Fix any $\varepsilon > 0$ and $f \in L^2(\Omega)$. Define $R_S(f, \varepsilon) = R_W(f, \varepsilon A/B)$, and call this quantity R. Then there are scalars $c_{\lambda,k}(\alpha)$ such that

$$\forall \, \alpha \in \mathbf{R}^d, \qquad \biggl\| M_\alpha f - \sum_{k=1}^N \sum_{\lambda \in \Lambda_k \cap Q_R(\alpha)} c_{\lambda,k}(\alpha) \, \tilde{g}_{\lambda,k} \biggr\|_2 \;<\; \frac{\varepsilon A}{B}.$$

Since $M_\alpha f = \sum_{k=1}^N \sum_{\lambda\in\Lambda_k} \langle M_\alpha f, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k}$, it follows that

$$\begin{aligned}
&\Bigl\| M_\alpha f - \sum_{k=1}^N \sum_{\lambda\in\Lambda_k\cap Q_R(\alpha)} \langle M_\alpha f, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k}\Bigr\|_2^2 \\
&\quad = \Bigl\| \sum_{k=1}^N \sum_{\lambda\in\Lambda_k\setminus Q_R(\alpha)} \langle M_\alpha f, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k}\Bigr\|_2^2 \\
&\quad \le \frac{1}{A} \sum_{k=1}^N \sum_{\lambda\in\Lambda_k\setminus Q_R(\alpha)} |\langle M_\alpha f, M_\lambda g_k\rangle|^2 \\
&\quad \le \frac{1}{A} \biggl(\sum_{k=1}^N \sum_{\lambda\in\Lambda_k\setminus Q_R(\alpha)} |\langle M_\alpha f, M_\lambda g_k\rangle|^2 \\
&\qquad\qquad + \sum_{k=1}^N \sum_{\lambda\in\Lambda_k\cap Q_R(\alpha)} |\langle M_\alpha f, M_\lambda g_k\rangle - c_{\lambda,k}(\alpha)|^2 \biggr) \\
&\quad \le \frac{B}{A} \Bigl\| \sum_{k=1}^N \sum_{\lambda\in\Lambda_k\setminus Q_R(\alpha)} \langle M_\alpha f, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k} \\
&\qquad\qquad + \sum_{k=1}^N \sum_{\lambda\in\Lambda_k\cap Q_R(\alpha)} (\langle M_\alpha f, M_\lambda g_k\rangle - c_{\lambda,k}(\alpha))\, \tilde{g}_{\lambda,k} \Bigr\|_2^2 \\
&\quad = \frac{B}{A} \Bigl\| M_\alpha f - \sum_{k=1}^N \sum_{\lambda\in\Lambda_k\cap Q_R(\alpha)} c_{\lambda,k}(\alpha)\, \tilde{g}_{\lambda,k} \Bigr\|_2^2 \; < \; \varepsilon.
\end{aligned}$$

Therefore $\mathcal{E}$ satisfies the Strong HAP. □

The next lemma shows that, assuming appropriate hypotheses, in order to establish that the HAP holds, it suffices to check that the relevant condition (equation (3.2) or (3.3)) holds for a complete subset of $L^2(\Omega)$.

Lemma 3.6. *Let Ω be a bounded subset of $\mathbf{R}^d$. Let $g_1, \dots, g_N \in L^2(\Omega)$ and $\Lambda_1, \dots, \Lambda_N \subset \mathbf{R}^d$ be such that $\mathcal{E} = \bigcup_{k=1}^N \mathcal{E}(g_k, \Lambda_k) = \bigcup_{k=1}^N \{M_\lambda g_k\}_{\lambda\in\Lambda_k}$ is a frame or an exact sequence for $L^2(\Omega)$. Let $\tilde{\mathcal{E}} = \bigcup_{k=1}^N \{\tilde{g}_{\lambda,k}\}_{\lambda\in\Lambda_k}$ be either any dual sequence (if $\mathcal{E}$ is a frame) or the biorthogonal system in $L^2(\Omega)$ (if $\mathcal{E}$ is exact). Then the following statements hold.*

(a) *$H_0 = \{f \in L^2(\Omega) :$ equation (3.2) holds$\}$ is a closed subspace of $L^2(\Omega)$.*

(b) *If we assume that* $\mathcal{E}$ *is a frame and* $\tilde{\mathcal{E}}$ *is its canonical dual frame, then* $H_1 = \{f \in L^2(\Omega) :$ *equation* (3.3) *holds*$\}$ *is a closed subspace of* $L^2(\Omega)$.

PROOF. (a) Assume that $f_1,\ f_2 \in H_0$ and $a_1,\ a_2 \in \mathbf{C}$ are given. Let $h = a_1 f_1 + a_2 f_2$. Choose any $\varepsilon > 0$, and set $R_i = R_W\bigl(f_i, \frac{\varepsilon}{2|a_i|}\bigr)$ for $i = 1,\ 2$. Define $R_W(h, \varepsilon) = \max\{R_1, R_2\}$, and call this quantity R.

Choose any $\alpha \in \mathbf{R}^d$. For $i = 1,\ 2$ we have $R_i \le R$, so $W(R_i, \alpha) \subset W(R, \alpha)$. Therefore, since (3.2) holds,

$$\operatorname{dist}\bigl(M_\alpha f_i, W(R,\alpha)\bigr) \;\le\; \operatorname{dist}\bigl(M_\alpha f_i, W(R_i,\alpha)\bigr) \;<\; \frac{\varepsilon}{2|a_i|}, \qquad i = 1, 2.$$

Let P_W denote the orthogonal projection onto $W(R, \alpha)$. Then

$$\begin{aligned}
\operatorname{dist}\bigl(M_\alpha h, W(R,\alpha)\bigr) &= \|(a_1 M_\alpha f_1 + a_2 M_\alpha f_2) - P_W(a_1 M_\alpha f_1 + a_2 M_\alpha f_2)\|_2 \\
&\le |a_1|\,\|M_\alpha f_1 - P_W(M_\alpha f_1)\|_2 + |a_2|\,\|M_\alpha f_2 - P_W(M_\alpha f_2)\|_2 \\
&= |a_1| \operatorname{dist}\bigl(M_\alpha f_1, W(R,\alpha)\bigr) + |a_2| \operatorname{dist}\bigl(M_\alpha f_2, W(R,\alpha)\bigr) \\
&< |a_1| \frac{\varepsilon}{2|a_1|} + |a_2| \frac{\varepsilon}{2|a_2|} \;=\; \varepsilon.
\end{aligned}$$

Thus $h \in H_0$, so H_0 is a linear subspace of $L^2(\Omega)$.

To show that H_0 is closed, suppose that f lies in the closure of H_0. Choose any $\varepsilon > 0$. Then there exists $h \in H_0$ such that $\|f - h\|_2 < \varepsilon/3$. Define $R_W(f, \varepsilon) = R_W(h, \varepsilon/3)$, and denote this quantity by R. Then for any $\alpha \in \mathbf{R}^d$, we have

$$\begin{aligned}
&\operatorname{dist}\bigl(M_\alpha f, W(R,\alpha)\bigr) \\
&\quad= \|M_\alpha f - P_W(M_\alpha f)\|_2 \\
&\quad\le \|M_\alpha f - M_\alpha h\|_2 + \|M_\alpha h - P_W(M_\alpha h)\|_2 + \|P_W(M_\alpha h - M_\alpha f)\|_2 \\
&\quad\le \|f - h\|_2 + \operatorname{dist}\bigl(M_\alpha h, W(R,\alpha)\bigr) + \|M_\alpha h - M_\alpha f\|_2 \\
&\quad< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \;=\; \varepsilon.
\end{aligned}$$

Therefore $f \in H_0$, so H_0 is closed.

(b) Let A, B be frame bounds for $\mathcal{E}$, so that $\frac{1}{B}$, $\frac{1}{A}$ are frame bounds for $\tilde{\mathcal{E}}$. Let S denote the frame operator for $\mathcal{E}$, so we have $S^{-1}(M_\lambda g_k) = \tilde{g}_{\lambda,k}$.

Assume that $f_1,\ f_2 \in H_1$ and $a_1,\ a_2 \in \mathbf{C}$ are given. Let $h = a_1 f_1 + a_2 f_2$. Choose any $\varepsilon > 0$, and set

$$R_i \;=\; R\biggl(f_i, \frac{\varepsilon A^{1/2}}{2B^{1/2} \|f_i\|^{1/2}\,|a_i|}\biggr), \qquad i = 1, 2,$$

and set $R = R_S(h,\varepsilon) = \max\{R_1, R_2\}$.

Choose any $\alpha \in \mathbf{R}^d$. For $i = 1, 2$ we have $R_i \le R$, so $\Lambda_k \backslash Q_R(\alpha) \subset \Lambda_k \backslash Q_{R_i}(\alpha)$. Therefore, for $i = 1, 2$ we have

$$\begin{aligned}
\Big\| M_\alpha f_i - \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \cap Q_R(\alpha)} \langle M_\alpha f_i, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k} \Big\|_2^2
&= \Big\| \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \backslash Q_R(\alpha)} \langle M_\alpha f_i, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k} \Big\|_2^2 \\
&\le \frac{1}{A} \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \backslash Q_R(\alpha)} |\langle M_\alpha f_i, M_\lambda g_k\rangle|^2 \\
&\le \frac{1}{A} \Big\langle \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \backslash Q_{R_i}(\alpha)} \langle M_\alpha f_i, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k},\, S M_\alpha f_i \Big\rangle \\
&\le \frac{1}{A} \Big\| \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \backslash Q_{R_i}(\alpha)} \langle M_\alpha f_i, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k} \Big\|_2 \|S M_\alpha f_i\|_2 \\
&< \frac{1}{A}\, \frac{\varepsilon^2 A}{4B\|f_i\|_2\, |a_i|^2}\, B\, \|M_\alpha f_i\|_2 \;=\; \frac{\varepsilon^2}{4|a_i|^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Big\| M_\alpha h - \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \cap Q_R\alpha)} \langle M_\alpha h, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k} \Big\|_2
&\le |a_1| \Big\| M_\alpha f_1 - \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \cap Q_R(\alpha)} \langle M_\alpha f_1, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k} \Big\|_2 \\
&\qquad + |a_2| \Big\| M_\alpha f_2 - \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \cap Q_R(\alpha)} \langle M_\alpha f_2, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k} \Big\|_2 \\
&< |a_1|\, \frac{\varepsilon}{2|a_1|} + |a_2|\, \frac{\varepsilon}{2|a_2|} \;=\; \varepsilon.
\end{aligned}$$

Thus $h \in H_1$, so H_1 is a linear subspace of $L^2(\Omega)$.

Next, to show that H_1 is closed, suppose that f lies in the closure of H_1. Choose any $\varepsilon > 0$. Then there exists $h \in H_1$ such that $\|f - h\|_2 < (\varepsilon A^{1/2})/(3B^{1/2})$. Set $R_S(f,\varepsilon) = R_S(h, \varepsilon/3)$, and denote this quantity by R. Then for any $\alpha \in \mathbf{R}^d$, we have

$$\begin{aligned}
\Big\| M_\alpha f - \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \cap Q_R(\alpha)} \langle M_\alpha f, M_\lambda g_k \rangle \, \tilde{g}_{\lambda,k} \Big\|_2 \\
&\le \|M_\alpha f - M_\alpha h\|_2 + \Big\| M_\alpha h - \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \cap Q_R(\alpha)} \langle M_\alpha h, M_\lambda g_k \rangle \, \tilde{g}_{\lambda,k} \Big\|_2 \\
&\qquad + \Big\| \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \cap Q_R(\alpha)} \langle M_\alpha f - M_\alpha h, M_\lambda g_k \rangle \, \tilde{g}_{\lambda,k} \Big\|_2 \\
&< \frac{\varepsilon A^{1/2}}{3B^{1/2}} + \frac{\varepsilon}{3} + \bigg(\frac{1}{A} \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \cap Q_R(\alpha)} |\langle M_\alpha f - M_\alpha h, M_\lambda g_k \rangle|^2 \bigg)^{1/2} \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \Big(\frac{B}{A}\Big)^{1/2} \|M_\alpha f - M_\alpha h\|_2 \;<\; \varepsilon,
\end{aligned}$$

where the second inequality follows from (2.3) and the third inequality from the definition of frame. Therefore $f \in H_1$, so H_1 is closed. □

3.3. **The HAP for Frames of Windowed Exponentials.** Now we establish that every frame of windowed exponentials satisfies the Strong HAP with respect to its canonical dual frame and the Weak HAP with respect to *any* dual frame.

Theorem 3.7. *Let Ω be a bounded subset of $\mathbf{R}^d$. Let $g_1, \dots, g_N \in L^2(\Omega)$ and $\Lambda_1, \dots, \Lambda_N \subset \mathbf{R}^d$ be given. Set $\Lambda = \bigcup_{k=1}^N \Lambda_k$, and assume $\mathcal{E} = \bigcup_{k=1}^N \mathcal{E}(g_k, \Lambda_k)$ is a frame for $L^2(\Omega)$.*

(a) *If $\tilde{\mathcal{E}} = \bigcup_{k=1}^N \{\tilde{g}_{\lambda,k}\}_{\lambda \in \Lambda_k}$ is any dual frame, then $\mathcal{E}$ satisfies the Weak HAP with respect to $\tilde{\mathcal{E}}$.*

(b) *If $\tilde{\mathcal{E}} = \bigcup_{k=1}^N \{\tilde{g}_{\lambda,k}\}_{\lambda \in \Lambda_k}$ is the canonical dual frame, then $\mathcal{E}$ satisfies the Strong HAP with respect to $\tilde{\mathcal{E}}$.*

PROOF. (a) Let A, B be frame bounds for $\mathcal{E}$, and let C, D be frame bounds for $\tilde{\mathcal{E}}$. By Lemma 3.2, we have $D^+(\Lambda) < \infty$, so by passing to subsequences if necessary we may assume that each Λ_k is δ_k-uniformly separated for some $\delta_k > 0$. Let $\delta = \min\{\delta_1/2, \dots, \delta_N/2\}$. Then any cube $Q_\delta(x)$ can contain at most one point of each Λ_k.

By Lemma 3.6(a), it suffices to show that equation (3.2) holds for a complete subset of $L^2(\Omega)$. We claim that (3.2) holds for the particular complete set $\{M_\omega \chi\}_{\omega \in \mathbf{R}^d}$, where $\chi = \chi_\Omega$. For simplicity, we will show below only that (3.2) holds for the particular function $\chi = M_0\chi$, but entirely similar calculations show that it holds for each function $M_\omega\chi$.

Fix any $\varepsilon > 0$. By Proposition 2.12, we have each $g_k \in W(\mathcal{C}, \ell^2)$, so we can find $M \in \mathbf{N}$ large enough that

$$\sum_{j\in\mathbf{Z}^d\setminus Q_M(0)} \sup_{\xi\in Q_\delta(j\delta)} |\hat{g}_k(\xi)| \;\le\; \frac{\varepsilon^2}{DN}, \qquad k = 1,\dots,N.$$

Set $R = R_S(\chi,\varepsilon) = (2M+1)\delta$.

Fix any $\alpha \in \mathbf{R}^d$. The frame expansion of $M_\alpha\chi$ is

$$M_\alpha\chi \;=\; \sum_{k=1}^{N}\sum_{\lambda\in\Lambda_k} \langle M_\alpha\chi, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k}.$$

If $\lambda \notin Q_R(\alpha)$, then there exists a unique $j \in \mathbf{Z}^d\setminus Q_M(0)$ such that $\lambda-\alpha \in Q_\delta(j\delta)$. Therefore,

$$\begin{aligned}
(3.4)\qquad &\Bigl\|M_\alpha\chi \;-\; \sum_{k=1}^{N}\sum_{\lambda\in\Lambda_k\cap Q_R(\alpha)} \langle M_\alpha\chi, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k}\Bigr\|_2^2\\
&\quad= \Bigl\|\sum_{k=1}^{N}\sum_{\lambda\in\Lambda_k\setminus Q_R(\alpha)} \langle M_\alpha\chi, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k}\Bigr\|_2^2\\
&\quad\le D\sum_{k=1}^{N}\sum_{\lambda\in\Lambda_k\setminus Q_R(\alpha)} |\langle M_\alpha\chi, M_\lambda g_k\rangle|^2\\
&\quad= D\sum_{k=1}^{N}\sum_{\lambda\in\Lambda_k\setminus Q_R(\alpha)} |\hat{g}_k(\alpha-\lambda)|^2\\
&\quad\le D\sum_{k=1}^{N}\sum_{j\in\mathbf{Z}^d\setminus Q_M(0)} \sup_{\xi\in Q_\delta(j\delta)} |\hat{g}_k(\xi)|^2\\
&\quad< DN\frac{\varepsilon^2}{DN} \;=\; \varepsilon^2,
\end{aligned}$$

where the first inequality follows from (2.3). Therefore equation (3.2) holds for the function χ.

(b) The proof is identical, except we observe that the calculation in (3.4) shows that equation (3.3) holds for the function χ, and hence it follows from Lemma 3.6(b) that the Strong HAP is satisfied. □

3.4. **The Weak HAP for Schauder Bases of Windowed Exponentials.** In this section we show that at least some Schauder bases of windowed exponentials possess the Weak HAP.

Theorem 3.8. *Let* Ω *be a bounded subset of* $\mathbf{R}^d$. *Let* $g_1, \dots, g_N \in L^2(\Omega)$ *and* $\Lambda_1, \dots, \Lambda_N \subset \mathbf{R}^d$ *be given. Set* $\Lambda = \bigcup_{k=1}^N \Lambda_k$. *Let* $\mathcal{E} = \bigcup_{k=1}^N \mathcal{E}(g_k, \Lambda_k)$ *be a Schauder basis for* $L^2(\Omega)$ *and let* $\tilde{\mathcal{E}} = \bigcup_{k=1}^N \{\tilde{g}_{\lambda,k}\}_{\lambda \in \Lambda_k}$ *be its dual basis. If either:*

(a) $\mathcal{E}$ *possesses a lower frame bound, or*

(b) $\hat{g}_k \in L^1(\mathbf{R}^d)$ *for* $k = 1, \dots, N$,

then $\mathcal{E}$ *satisfies the Weak HAP (with respect to its dual basis).*

Proof. (a) Suppose that $\mathcal{E}$ has a lower frame bound A. Then by Lemma 2.4, the dual basis $\tilde{\mathcal{E}}$ is a Bessel sequence with Bessel bound $D = 1/A$. The proof then proceeds almost identically to the proof of Theorem 3.7(a), so will be omitted.

(b) Since $\mathcal{E}$ is a bounded basis, its dual basis $\tilde{\mathcal{E}}$ is also a bounded basis. Hence $D = \sup \|\tilde{g}_{\lambda,k}\|_2 < \infty$. We have $D^+(\Lambda) < \infty$ by Lemma 3.1. Let δ be as in the proof of Theorem 3.7(a). In light of Lemma 3.6(a), we need only show that equation (3.2) holds for each of the functions $M_\omega \chi$ with $\omega \in \mathbf{R}^d$. For simplicity, we present only the case $\omega = 0$.

Fix any $\varepsilon > 0$. Since we have assumed $\hat{g}_k \in L^1(\mathbf{R}^d)$, it follows from Proposition 2.12 that $\hat{g}_k \in W(\mathcal{C}, \ell^1)$. Therefore, we can find M large enough that

$$\sum_{j \in \mathbf{Z}^d \setminus Q_M(0)} \sup_{\xi \in Q_\delta(j\delta)} |\hat{g}_k(\xi)| \le \frac{\varepsilon}{DN}, \qquad k = 1, \dots, N.$$

Set $R = R_W(\chi, \varepsilon) = (2M+1)\delta$.

Fix now any $\alpha \in \mathbf{R}^d$. Since $\mathcal{E}$ is a Schauder basis with dual basis $\tilde{\mathcal{E}}$, and since $\Lambda_k \cap Q_R(\alpha)$ is a finite set, we can write

$$\begin{aligned} &\Big\| M_\alpha \chi - \sum_{k=1}^N \sum_{\lambda \in \Lambda_k \cap Q_R(\alpha)} \langle M_\alpha \chi, M_\lambda g_k \rangle \, \tilde{g}_{\lambda,k} \Big\|_2 \\ &\qquad = \Big\| \sum_{k=1}^N \sum_{\lambda \in \Lambda_k \setminus Q_R(\alpha)} \langle M_\alpha \chi, M_\lambda g_k \rangle \, \tilde{g}_{\lambda,k} \Big\|_2, \end{aligned}$$

with respect to some appropriate ordering of these series. If $\lambda \notin Q_R(\alpha)$, then there exists a unique $j \in \mathbf{Z}^d \setminus Q_M(0)$ such that $\lambda - \alpha \in Q_\delta(j\delta)$. Applying the

Triangle Inequality, we therefore compute that

$$\begin{aligned}
\operatorname{dist}(M_\alpha\chi, W(R,\alpha)) \;&\le\; \Bigl\| M_\alpha\chi \;-\; \sum_{k=1}^{N} \sum_{\lambda\in\Lambda_k\cap Q_R(\alpha)} \langle M_\alpha\chi, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k} \Bigr\|_2 \\
&=\; \Bigl\| \sum_{k=1}^{N} \sum_{\lambda\in\Lambda_k\setminus Q_R(\alpha)} \langle M_\alpha\chi, M_\lambda g_k\rangle\, \tilde{g}_{\lambda,k} \Bigr\|_2 \\
&\le\; \Bigl(\sup \|\tilde{g}_{\lambda,k}\|_2\Bigr) \sum_{k=1}^{N} \sum_{\lambda\in\Lambda_k\setminus Q_R(\alpha)} |\langle M_\alpha\chi, M_\lambda g_k\rangle| \\
&\le\; D \sum_{k=1}^{N} \sum_{\lambda\in\Lambda_k\setminus Q_R(\alpha)} |\hat{g}_k(\alpha-\lambda)| \\
&\le\; D \sum_{k=1}^{N} \sum_{j\in\mathbf{Z}^d\setminus Q_M(0)} \sup_{\xi\in Q_\delta(j\delta)} |\hat{g}_k(\xi)| \\
&<\; DN\frac{\varepsilon}{DN} \;=\; \varepsilon.
\end{aligned}$$

□

3.5. **The Comparison Theorem.** We saw in Theorem 3.7 that all frames of windowed exponentials possess the Strong HAP, and gave in Theorem 3.8 some sufficient conditions under which a Schauder basis will possess the Weak HAP. We do not have sufficient conditions under which an exact sequence will possess either HAP, but we show in this section that if a systems of windowed exponentials is merely exact and satisfies the Weak HAP, then certain density conditions must be satisfied in comparison to any other Schauder basic sequence of windowed exponentials. This Comparison Theorem is directly inspired by the double-projection idea of Ramanathan and Steger [28].

Theorem 3.9 (Comparison Theorem)**.** *Let Ω be a bounded subset of $\mathbf{R}^d$. Assume that*

(a) $g_1,\dots,g_N \in L^2(\Omega)$ *and* $\Lambda_1,\dots,\Lambda_N \subset \mathbf{R}^d$ *are such that*

$$\mathcal{E} \;=\; \bigcup_{k=1}^{N} \mathcal{E}(g_k,\Lambda_k)$$

is either

(i) *a frame, or*

(ii) *an exact sequence which possesses the Weak HAP,*

(b) $\phi_1, \dots, \phi_M \in L^2(\Omega)$ *and* $\Delta_1, \dots, \Delta_M \subset \mathbf{R}^d$ *are such that*

$$\Phi \;=\; \bigcup_{k=1}^{M} \mathcal{E}(\phi_k, \Delta_k)$$

is a Schauder basic sequence in $L^2(\Omega)$.

Set $\Lambda = \bigcup_{k=1}^N \Lambda_k$ *and* $\Delta = \bigcup_{k=1}^M \Delta_k$. *Then*

$$D^-(\Delta) \;\le\; D^-(\Lambda) \qquad \textit{and} \qquad D^+(\Delta) \;\le\; D^+(\Lambda).$$

Proof. Let $\tilde{\mathcal{E}} = \bigcup_{k=1}^N \{\tilde{g}_{\lambda,k}\}_{\lambda \in \Lambda_k}$ denote either the canonical dual frame of $\mathcal{E}$ (if hypothesis (i) applies) or the sequence biorthogonal to $\mathcal{E}$ (if hypothesis (ii) applies). In either case, by hypothesis or by Theorem 3.7, $\mathcal{E}$ possesses the Weak HAP.

We are given that Φ is a Schauder basis for its closed span within $L^2(\Omega)$. Let $\tilde{\Phi} = \bigcup_{k=1}^M \{\tilde{\phi}_{\gamma,k}\}_{\gamma \in \Delta_k}$ denote the dual basis within that closed span.

Let $W(h,\alpha)$ be as in (3.1), and set

$$V(h,\beta) \;=\; \operatorname{span}\bigl\{M_\alpha \phi_k : \alpha \in \Delta_k \cap Q_h(\beta),\ k = 1, \dots, M\bigr\}.$$

Note that $V(h,\beta)$ is finite-dimensional since Δ has finite density by Lemma 3.1.

Fix any $\varepsilon > 0$. Applying the definition of the Weak HAP to the functions $f = \phi_k$, we see that there exists an $R > 0$ such that

$$\forall\, \alpha \in \mathbf{R}^d, \quad \operatorname{dist}\bigl(M_\alpha \phi_k,\, W(R,\alpha)\bigr) \;<\; \frac{\varepsilon}{D}, \qquad k = 1, \dots, M, \tag{3.5}$$

where

$$D \;=\; \sup\bigl\{\|\tilde{\phi}_{\gamma,k}\| : \gamma \in \Delta_k,\ k = 1, \dots, M\bigr\}.$$

Fix any $h > 0$ and $\beta \in \mathbf{R}^d$. Let P_V and P_W denote the orthogonal projections of $L^2(\Omega)$ onto $V = V(h,\beta)$ and $W = W(R+h,\beta)$, respectively. Define $T\colon V \to V$ by $T = P_V P_W = P_V P_W P_V$. Note that T is self-adjoint and V is finite-dimensional, so T has a finite, real trace.

We will estimate the trace of T. First note that every eigenvalue λ of T satisfies $|\lambda| \le \|T\| \le \|P_V\|\,\|P_W\| = 1$. Since the trace is the sum of the eigenvalues, this provides us with an upper bound for the trace of T:

$$\operatorname{trace}(T) \;\le\; \operatorname{rank}(T) \;\le\; \dim(W) \;=\; \#\bigl(\Lambda \cap Q_{R+h}(\beta)\bigr). \tag{3.6}$$

For a lower estimate, note that $\{M_\alpha \phi_k : \alpha \in \Delta_k \cap Q_h(\beta),\ k = 1, \dots, M\}$ is a basis for V. The dual basis in V is $\{P_V \tilde{\phi}_{\alpha,k} : \alpha \in \Delta_k \cap Q_h(\beta),\ k = 1, \dots, M\}$.

Therefore

$$\text{(3.7)} \qquad \begin{aligned} \operatorname{trace}(T) &= \sum_{k=1}^{M} \sum_{\alpha \in \Delta_k \cap Q_h(\beta)} \bigl\langle T(M_\alpha \phi_k),\, P_V \tilde{\phi}_{\alpha,k} \bigr\rangle \\ &= \sum_{k=1}^{M} \sum_{\alpha \in \Delta_k \cap Q_h(\beta)} \bigl\langle P_W(M_\alpha \phi_k),\, P_V \tilde{\phi}_{\alpha,k} \bigr\rangle \\ &= \sum_{k=1}^{M} \sum_{\alpha \in \Delta_k \cap Q_h(\beta)} \Bigl(\bigl\langle M_\alpha \phi_k,\, P_V \tilde{\phi}_{\alpha,k} \bigr\rangle \\ &\qquad\qquad + \bigl\langle (P_W - \mathbf{1})(M_\alpha \phi_k),\, P_V \tilde{\phi}_{\alpha,k} \bigr\rangle \Bigr). \end{aligned}$$

However,

$$\text{(3.8)} \qquad \bigl\langle M_\alpha \phi_k,\, P_V \tilde{\phi}_{\alpha,k} \bigr\rangle \;=\; \bigl\langle P_V(M_\alpha \phi_k),\, \tilde{\phi}_{\alpha,k} \bigr\rangle \;=\; \bigl\langle M_\alpha \phi_k,\, \tilde{\phi}_{\alpha,k} \bigr\rangle \;=\; 1,$$

the last equality following from the biorthogonality of Φ and $\tilde{\Phi}$. Additionally, if $\alpha \in Q_h(\beta)$ then we have $Q_R(\alpha) \subset Q_{R+h}(\beta)$, so $W(R,\alpha) \subset W(R+h,\beta)$ and therefore

$$\text{(3.9)} \qquad \begin{aligned} \bigl|\bigl\langle (P_W - \mathbf{1})(M_\alpha \phi_k),\, P_V \tilde{\phi}_{\alpha,k} \bigr\rangle\bigr| &\le \|(P_W - \mathbf{1})(M_\alpha \phi_k)\|_2 \, \|P_V \tilde{\phi}_{\alpha,k}\| \\ &\le \operatorname{dist}\bigl(M_\alpha \phi_k, W(R+h,\beta)\bigr) \, \|\tilde{\phi}_{\alpha,k}\|_2 \\ &\le \operatorname{dist}\bigl(M_\alpha \phi_k, W(R,\alpha)\bigr) \, D \\ &\le \frac{\varepsilon}{D} \, D \;=\; \varepsilon. \end{aligned}$$

Combining (3.7)–(3.9) yields the lower bound

$$\text{(3.10)} \qquad \operatorname{trace}(T) \;\ge\; \sum_{k=1}^{M} \sum_{\alpha \in \Delta_k \cap Q_h(\beta)} (1-\varepsilon) \;=\; (1-\varepsilon)\, \#\bigl(\Delta \cap Q_h(\beta)\bigr).$$

Finally, combining the upper estimate (3.6) with the lower estimate (3.10), we see that

$$\forall\, \beta \in \mathbf{R}^d, \quad \forall\, h > 0, \quad \#\bigl(\Lambda \cap Q_{R+h}(\beta)\bigr) \;\ge\; (1-\varepsilon)\, \#\bigl(\Delta \cap Q_h(\beta)\bigr),$$

and so

$$\begin{aligned} D^-(\Delta) &= \liminf_{h\to\infty} \inf_{\beta \in \mathbf{R}^d} \frac{\#\bigl(\Delta \cap Q_h(\beta)\bigr)}{h^d} \\ &\le \frac{1}{1-\varepsilon} \liminf_{h\to\infty} \inf_{\beta \in \mathbf{R}^d} \frac{\#\bigl(\Lambda \cap Q_{R+h}(\beta)\bigr)}{(R+h)^d} \, \frac{(R+h)^d}{h^d} \;=\; \frac{1}{1-\varepsilon}\, D^-(\Lambda). \end{aligned}$$

Since ε is arbitrary, we conclude that $D^-(\Delta) \le D^-(\Lambda)$. A similar calculation shows that $D^+(\Delta) \le D^+(\Lambda)$. □

3.6. **Density and Bounds for Frames of Windowed Exponentials.** Combining our previous results on the HAP and the Comparison Theorem, we obtain the following necessary density conditions.

Theorem 3.10. *Let Ω be a bounded subset of $\mathbf{R}^d$ such that $|\partial\Omega| = 0$. Let $g_1, \dots, g_N \in L^2(\Omega) \setminus \{0\}$ and $\Lambda_1, \dots, \Lambda_N \subset \mathbf{R}^d$ be given. Set $\mathcal{E} = \bigcup_{k=1}^N \mathcal{E}(g_k, \Lambda_k)$ and $\Lambda = \bigcup_{k=1}^N \Lambda_k$.*

(a) *If $\mathcal{E}$ is a frame for $L^2(\Omega)$, then $|\Omega| \le D^-(\Lambda) \le D^+(\Lambda) < \infty$.*

(b) *If $\mathcal{E}$ is a Riesz sequence in $L^2(\Omega)$, then $0 \le D^-(\Lambda) \le D^+(\Lambda) \le |\Omega|$.*

(c) *If $\mathcal{E}$ is a Riesz basis for $L^2(\Omega)$, then $D^-(\Lambda) = D^+(\Lambda) = |\Omega|$.*

Proof. (a) Suppose $\mathcal{E}$ is a frame for $L^2(\Omega)$. By Theorem 3.7, we know that $\mathcal{E}$ possesses the Strong HAP, and furthermore $D^+(\Lambda) < \infty$ by Lemma 3.2. By Lemma 2.10(a), given $\varepsilon > 0$ we know there exist functions $\phi_1, \dots, \phi_M \in L^2(\Omega)$ and $\Delta_1, \dots, \Delta_M \subset \mathbf{R}^d$ such that $\Phi = \bigcup_{k=1}^M \mathcal{E}(\phi_k, \Delta_k)$ is an orthonormal sequence in $L^2(\Omega)$, and furthermore, $|\Omega| - \varepsilon \le D^\pm(\Delta) \le |\Omega|$, where $\Delta = \bigcup_{k=1}^M \Delta_k$. Applying Theorem 3.9 to $\mathcal{E}$ and Φ, we conclude that $|\Omega| - \varepsilon \le D^-(\Delta) \le D^-(\Lambda)$. Since ε is arbitrary, the result follows.

(b) Suppose that $\mathcal{E}$ is a Riesz sequence in $L^2(\Omega)$. By Lemma 2.10(b), given $\varepsilon > 0$ there exist functions $\phi_1, \dots, \phi_M \in L^2(\Omega)$ and $\Delta_1, \dots, \Delta_M \subset \mathbf{R}^d$ such that $\Phi = \bigcup_{k=1}^M \mathcal{E}(\phi_k, \Delta_k)$ is a frame for $L^2(\Omega)$, and furthermore, $|\Omega| \le D^\pm(\Delta) \le |\Omega| + \varepsilon$, where $\Delta = \bigcup_{k=1}^M \Delta_k$. Since Φ possesses the Strong HAP, by applying Theorem 3.9 to Φ and $\mathcal{E}$, we conclude that $D^+(\Lambda) \le D^+(\Delta) \le |\Omega| + \varepsilon$. Since ε is arbitrary, the result follows.

(c) Every Riesz basis is both a frame and a Riesz sequence, so the result follows by combining statements (a) and (b). □

We now derive new relationships between the density, frame bounds, and norms of the generators of a frame of windowed exponentials. In this result, the fact that the elements $M_\lambda g_k$ of $\bigcup_{k=1}^N \mathcal{E}(g_k, \Lambda_k)$ can have different norms requires us to define a generalization of Beurling density where the norm of the element associated to each point is taken into account.

Definition 3.11. Let Ω be a bounded subset of $\mathbf{R}^d$. Then given $g_1, \dots, g_N \in L^2(\Omega)$ and $\Lambda_1, \dots, \Lambda_N \subset \mathbf{R}^d$, we define the *weighted lower Beurling density* with respect to $g_1, \dots, g_N$ and $\Lambda_1, \dots, \Lambda_N$ to be

$$D_W^-(g_1, \dots, g_N; \Lambda_1, \dots, \Lambda_N) = \left(\frac{1}{N}\sum_{k=1}^N \|g_k\|_2^2\right)^{-1} \liminf_{h\to\infty} \inf_{\beta\in\mathbf{R}^d} \frac{\sum_{k=1}^N \#\big(\Lambda_k \cap Q_h(\beta)\big)\, \|g_k\|_2^2}{h^d}.$$

We also make an analogous definition of $D_W^+(g_1, \dots, g_N; \Lambda_1, \dots, \Lambda_N)$.

Obviously, if $\|g_1\|_2 = \cdots = \|g_N\|_2$, then $D_W^\pm(g_1, \dots, g_N; \Lambda_1, \dots, \Lambda_N) = D^\pm(\Lambda)$, where $\Lambda = \bigcup_{k=1}^N \Lambda_k$.

Theorem 3.12. *Let Ω be a bounded subset of $\mathbf{R}^d$. Let $g_1, \dots, g_N \in L^2(\Omega)$ and $\Lambda_1, \dots, \Lambda_N \subset \mathbf{R}^d$ be such that $\mathcal{E} = \bigcup_{k=1}^N \mathcal{E}(g_k, \Lambda_k)$ is a frame for $L^2(\Omega)$ with frame bounds A, B. Define $\Lambda = \bigcup_{k=1}^N \Lambda_k$. Then the following statements hold.*

(a) *We have*

$$\begin{aligned} A|\Omega| \ &\le\ D_W^-(g_1, \dots, g_N; \Lambda_1, \dots, \Lambda_N)\, \frac{1}{N}\sum_{k=1}^N \|g_k\|_2^2 \\ &\le\ D_W^+(g_1, \dots, g_N; \Lambda_1, \dots, \Lambda_N)\, \frac{1}{N}\sum_{k=1}^N \|g_k\|_2^2 \ \le\ B|\Omega|. \end{aligned}$$

(b) *If $\|g_1\|_2^2 = \cdots = \|g_N\|_2^2 = \mathcal{N}_\mathcal{E}$, then*

$$A|\Omega| \ \le\ D^-(\Lambda)\, \mathcal{N}_\mathcal{E} \ \le\ D^+(\Lambda)\, \mathcal{N}_\mathcal{E} \ \le\ B|\Omega|.$$

(c) *If $\|g_1\|_2^2 = \cdots = \|g_N\|_2^2 = \mathcal{N}_\mathcal{E}$ and $\mathcal{E}$ is tight ($A = B$), then Λ has uniform Beurling density*

$$D^\pm(\Lambda) \ =\ \frac{A\,|\Omega|}{\mathcal{N}_\mathcal{E}}.$$

Proof. (a) For $\xi \in \mathbf{R}^d$, consider the function $f_\xi(x) = M_\xi \chi_\Omega(x) = e^{2\pi i \xi\cdot x}\, \chi_\Omega(x)$. By definition of frame,

$$A\, \|f_\xi\|_2^2 \ \le\ \sum_{k=1}^N \sum_{\lambda\in\Lambda_k} |\langle f_\xi, M_\lambda g_k\rangle|^2 \ \le\ B\, \|f_\xi\|_2^2. \tag{3.11}$$

Using the facts that $\|f_\xi\|_2^2 = |\Omega|$ and $\langle f_\xi, M_\lambda g_k\rangle = \overline{\hat{g}_k(\xi - \lambda)}$, if we integrate (3.11) over a cube $Q_h(x)$ we obtain

$$A|\Omega| h^d \ \le\ \sum_{k=1}^N \sum_{\lambda\in\Lambda_k} \int_{Q_h(x)} |\hat{g}_k(\xi-\lambda)|^2\, d\xi \ \le\ B|\Omega|h^d, \qquad x \in \mathbf{R}^d,\ h > 0.$$

Choose any $\varepsilon > 0$ and fix $r > 0$ large enough that

$$\int_{\mathbf{R}^d \setminus Q_r(0)} |\hat{g}_k(\xi)|^2 \, d\xi \;<\; \varepsilon, \qquad k = 1, \dots, N.$$

Since $D^+(\Lambda) < \infty$, as in the discussion following Definition 2.1, we can find a constant K such that for all $x \in \mathbf{R}^d$ and all h large enough we have both

$$\#\big(\Lambda \cap Q_h(x)\big) \;\le\; K\, h^d$$

and

$$\#\big(\Lambda \cap Q_{h+r}(x) \setminus Q_{h-r}(x)\big) \;\le\; K\,\big((h+r)^d - (h-r)^d\big).$$

We now make the decomposition

$$\sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k} \int_{Q_h(x)} |\hat{g}_k(\xi - \lambda)|^2 \, d\xi \;=\; I_1(x,h) - I_2(x,h) + I_3(x,h) + I_4(x,h),$$

where

$$\begin{aligned}
I_1(x,h) \;&=\; \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \cap Q_{h-r}(x)} \int_{\mathbf{R}^d} |\hat{g}_k(\xi - \lambda)|^2 \, d\xi, \\
I_2(x,h) \;&=\; \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \cap Q_{h-r}(x)} \int_{\mathbf{R}^d \setminus Q_h(x)} |\hat{g}_k(\xi - \lambda)|^2 \, d\xi, \\
I_3(x,h) \;&=\; \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \cap Q_{h+r}(x) \setminus Q_{h-r}(x)} \int_{Q_h(x)} |\hat{g}_k(\xi - \lambda)|^2 \, d\xi, \\
I_4(x,h) \;&=\; \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \setminus Q_{h+r}(x)} \int_{Q_h(x)} |\hat{g}_k(\xi - \lambda)|^2 \, d\xi.
\end{aligned}$$

We estimate each of these in turn. First,

$$I_1(x,h) \;=\; \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \cap Q_{h-r}(x)} \|\hat{g}_k\|_2^2 \;=\; \sum_{k=1}^{N} \|g_k\|_2^2 \; \#\big(\Lambda_k \cap Q_{h-r}(x)\big).$$

Trivially, $-I_2(x,h) \le 0$ always.

Every frame is bounded above in norm. In particular, $\|M_\lambda g_k\|_2^2 \le B$ for all $\lambda \in \Lambda_k$, $k = 1, \dots, N$. Therefore, for h large enough,

$$I_3(x,h) \;\le\; \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \cap Q_{h+r}(x) \setminus Q_{h-r}(x)} \|\hat{g}_k\|_2^2 \;\le\; B\,\#\bigl(\Lambda \cap Q_{h+r}(x) \setminus Q_{h-r}(x)\bigr)$$
$$\le\; BK\,\bigl((h+r)^d - (h-r)^d\bigr).$$

To estimate $I_4(x,h)$, note that if $\lambda \notin Q_{h+r}(x)$ and $\xi \in Q_h(x)$ then $\xi - \lambda \in Q_h(x-\lambda) \subset \mathbf{R}^d \setminus Q_r(0)$. Furthermore, each cube in $\{Q_h(x-\lambda)\}_{\lambda\in\Lambda}$ can intersect at most $K\,h^d$ of the others. Therefore, for h large enough,

$$\begin{aligned} I_4(x,h) \;&\le\; \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k \setminus Q_{h+r}(x)} \int_{Q_h(x-\lambda)} |\hat{g}_k(\xi)|^2\, d\xi \\ &\le\; \sum_{k=1}^{N} Kh^d \int_{\mathbf{R}^d \setminus Q_r(0)} |\hat{g}_k(\xi)|^2\, d\xi \;\le\; NKh^d\varepsilon. \end{aligned}$$

Combining the above estimates, we see that

$$\begin{aligned} A|\Omega|h^d \;&\le\; I_1(x,h) + 0 + I_3(x,h) + I_4(x,h) \\ &\le\; \left(\sum_{k=1}^{N} \|g_k\|_2^2\; \#\bigl(\Lambda_k \cap Q_{h-r}(x)\bigr)\right) \\ &\qquad +\; BK\,\bigl((h+r)^d - (h-r)^d\bigr) \;+\; NKh^d\varepsilon. \end{aligned}$$

Therefore

$$\begin{aligned} A|\Omega| \;&=\; \liminf_{h\to\infty}\, \inf_{x\in\mathbf{R}^d} \frac{A|\Omega|h^d}{h^d} \\ &\le\; \liminf_{h\to\infty}\, \inf_{x\in\mathbf{R}^d} \left(\sum_{k=1}^{N} \|g_k\|_2^2\; \frac{\#\bigl(\Lambda_k \cap Q_{h-r}(x)\bigr)}{(h-r)^d}\, \frac{(h-r)^d}{h^d}\right) + \\ &\qquad \limsup_{h\to\infty} \left(\frac{BK\,\bigl((h+r)^d - (h-r)^d\bigr)}{h^d} \;+\; \frac{NKh^d\varepsilon}{h^d}\right) \\ &=\; D_W^-(g_1,\dots,g_N;\Lambda_1,\dots,\Lambda_N)\, \frac{1}{N} \sum_{k=1}^{N} \|g_k\|_2^2 \;+\; 0 \;+\; NK\varepsilon. \end{aligned}$$

Since ε is arbitrary, we conclude that

$$A|\Omega| \;\le\; D_W^-(g_1,\dots,g_N;\Lambda_1,\dots,\Lambda_N)\, \frac{1}{N} \sum_{k=1}^{N} \|g_k\|_2^2,$$

and a similar calculation gives the upper estimate.

(b), (c) These are immediate consequences of part (a). □

For the case of a single generator, we obtain the following corollary.

Corollary 3.13. *Let Ω be a bounded subset of $\mathbf{R}^d$. Let $g \in L^2(\Omega)$ and $\Lambda \subset \mathbf{R}^d$ be such that $\mathcal{E}(g, \Lambda)$ is a frame for $L^2(\Omega)$ with frame bounds A, B. Then the following statements hold.*

(a) $A|\Omega| \;\le\; D^-(\Lambda)\, \|g\|_2^2 \;\le\; D^+(\Lambda)\, \|g\|_2^2 \;\le\; B|\Omega|$.

(b) *If $\mathcal{E}(g, \Lambda)$ is tight ($A = B$), then Λ has uniform Beurling density*

$$D^{\pm}(\Lambda) \;=\; \frac{A\,|\Omega|}{\|g\|_2^2}.$$

3.7. **Density for Schauder Bases and Exact Systems of Windowed Exponentials.** We now derive necessary conditions for the existence of Schauder bases and exact systems of windowed exponentials, but these are weaker than the results we obtained for frames. Sufficient conditions under which a Schauder basis of windowed exponentials will possess the Weak HAP were given in Theorem 3.7, but we do not have sufficient conditions for when an exact system will possess either the Weak HAP. However, if it is the case that the Weak HAP is satisfied, then we can prove the following necessary conditions.

Theorem 3.14. *Let Ω be a bounded subset of $\mathbf{R}^d$ such that $|\partial\Omega| = 0$. Let $g_1, \dots, g_N \in L^2(\Omega)$ and $\Lambda_1, \dots, \Lambda_N \subset \mathbf{R}^d$ be given, and set $\mathcal{E} = \bigcup_{k=1}^N \mathcal{E}(g_k, \Lambda_k)$ and $\Lambda = \bigcup_{k=1}^N \Lambda_k$.*

(a) *If $\mathcal{E}$ is an exact sequence in $L^2(\Omega)$ which possesses the Weak HAP, then*

$$|\Omega| \;\le\; D^-(\Lambda) \;\le\; D^+(\Lambda) \;\le\; \infty.$$

(b) *If $\mathcal{E}$ is a Schauder basic sequence in $L^2(\Omega)$, then*

$$0 \;\le\; D^-(\Lambda) \;\le\; D^+(\Lambda) \;\le\; |\Omega|.$$

(c) *If $\mathcal{E}$ is a Schauder basis for $L^2(\Omega)$ which possesses the Weak HAP, then*

$$D^-(\Lambda) \;=\; D^+(\Lambda) \;=\; |\Omega|.$$

Proof. (a) Suppose that $\mathcal{E}$ is an exact sequence which possesses the Weak HAP. Then by Lemma 2.10, given $\varepsilon > 0$ there exist functions $\phi_1, \dots, \phi_M \in L^2(\Omega)$ and $\Delta_1, \dots, \Delta_M \subset \mathbf{R}^d$ such that $\Phi = \bigcup_{k=1}^M \mathcal{E}(\phi_k, \Delta_k)$ is an orthonormal sequence in $L^2(\Omega)$, and furthermore, $|\Omega| - \varepsilon \le D^{\pm}(\Delta) \le |\Omega|$, where $\Delta = \bigcup_{k=1}^M \Delta_k$. Applying Theorem 3.9 to $\mathcal{E}$ and Φ, we conclude that $|\Omega| - \varepsilon \le D^-(\Delta) \le D^-(\Lambda)$. Since ε is arbitrary, the result follows.

(b) Suppose that $\mathcal{E}$ is a Schauder basic sequence in $L^2(\Omega)$. Then by Lemma 2.10, given $\varepsilon > 0$ there exist functions $\phi_1, \dots, \phi_M \in L^2(\Omega)$ and $\Delta_1, \dots, \Delta_M \subset \mathbf{R}^d$ such that $\Phi = \bigcup_{k=1}^M \mathcal{E}(\phi_k, \Delta_k)$ is a frame for $L^2(\Omega)$, and furthermore, $|\Omega| \le D^{\pm}(\Delta) \le |\Omega| + \varepsilon$, where $\Delta = \bigcup_{k=1}^M \Delta_k$. Applying Theorem 3.9 to Φ and $\mathcal{E}$, we conclude that $D^-(\Lambda) \le D^+(\Delta) \le |\Omega| + \varepsilon$. Since ε is arbitrary, the result follows.

(c) This follows by combining parts (a) and (b). □

We believe that Theorem 3.14(c) should apply to all Schauder bases of windowed exponentials.

Conjecture 3.15. *Let Ω be a bounded subset of $\mathbf{R}^d$. If $\mathcal{E} = \bigcup_{k=1}^N \mathcal{E}(g_k, \Lambda_k)$ is a Schauder basis for $L^2(\Omega)$, then $\mathcal{E}$ possesses the Weak HAP, and $D^-(\Lambda) = D^+(\Lambda) = |\Omega|$, where $\Lambda = \bigcup_{k=1}^N \Lambda_k$.*

The case for exact systems is less clear, so we close by asking whether analogous results must hold for exact systems. Namely, if $\mathcal{E} = \bigcup_{k=1}^N \mathcal{E}(g_k, \Lambda_k)$ is an exact system in $L^2(\Omega)$, must $\mathcal{E}$ possess the Weak or Strong HAP? Is it the case that $D^-(\Lambda) = D^+(\Lambda) = |\Omega|$?

Acknowledgments

Part of the research for this paper was performed while the second author was visiting the School of Mathematics at the Georgia Institute of Technology. This author thanks the School for its hospitality and support during this visit. The authors also thank Gerard Ascensi, Radu Balan, Pete Casazza, Ole Christensen, Baiqiao Deng, Mihail Kolountzakis, Zeph Landau, Jay Ramanathan, and David Walnut for valuable discussions.

References

[1] A. Aldroubi and K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, *SIAM Rev.*, **43** (2001), 585–620.
[2] K. I. Babenko, On conjugate functions (Russian), *Doklady Akad. Nauk SSSR (N. S.)*, **62** (1948), 157–160.
[3] J. J. Benedetto, A. Powell, and O. Yilmaz, Sigma-Delta quantization and finite frames, *IEEE Trans. Inform. Theory*, **52** (2006), 1990–2005.
[4] R. P. Boas and H. Pollard, The multiplicative completion of sets of functions, *Bull. Amer. Math. Soc.*, **54** (1948), 518–522.
[5] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2003.
[6] O. Christensen, B. Deng, and C. Heil, Density of Gabor frames, *Appl. Comput. Harmon. Anal.*, **7** (1999), 292–304.
[7] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.

[8] Y. C. Eldar and G. D. Forney, Optimal tight frames and quantum measurement, *IEEE Trans. Information Theory*, **48** (2002), 599–610.

[9] H. G. Feichtinger, Banach convolution algebras of Wiener type, in: *Functions, Series, Operators, Proc. Conf. Budapest* **38**, Colloq. Math. Soc. János Bolyai, 1980, pp. 509–524.

[10] H. G. Feichtinger, Banach spaces of distributions of Wiener's type and interpolation, in: *Functional Analysis and Approximation* (Oberwolfach, 1980), Internat. Ser. Numer. Math. **60**, Birkhäuser, Basel-Boston, 1981, pp. 153–165.

[11] H. G. Feichtinger and K. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions, I, *J. Funct. Anal.*, **86** (1989), 307–340.

[12] H. G. Feichtinger and K. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions, II, *Monatsh. Math.*, **108** (1989), 129–148.

[13] V. K. Goyal, J. Kovačević, and J. A. Kelner, *Appl. Comput. Harmon. Anal.*, **10** (2001), 203–233.

[14] R. Gribonval and M. Nielsen, Nonlinear approximation with dictionaries. I. Direct estimates, *J. Fourier Anal. Appl.*, **10** (2004), 51–71.

[15] K. Gröchenig and H. Razafinjatovo, On Landau's necessary density conditions for sampling and interpolation of band-limited functions, *J. London Math. Soc. (2)*, **54** (1996), 557–565.

[16] D. Han and D. R. Larson, Frames, bases and group representations *Mem. Amer. Math. Soc.*, **147**, No. 697 (2000).

[17] C. Heil, An introduction to weighted Wiener amalgams, in: *Wavelets and their Applications* (Chennai, January 2002), M. Krishna, R. Radha and S. Thangavelu, eds., Allied Publishers, New Delhi, 2003, pp. 183–216.

[18] K. S. Kazarian, The multiplicative completion of basic sequences in L^p, $1 \le p < \infty$, to bases in L^p (Russian), *Akad. Nauk Armjan. SSR Dokl.*, **62** (1976), 203–209.

[19] K. S. Kazarian, The multiplicative complementation of some incomplete orthonormal systems to bases in L^p, $1 \le p < \infty$ (Russian), *Anal. Math.*, **4** (1978), 37–52.

[20] K. S. Kazarian and R. E. Zink, Some ramifications of a theorem of Boas and Pollard concerning the completion of a set of functions in L^2, *Trans. Amer. Math. Soc.*, **349** (1997), 4367–4383.

[21] M. N. Kolountzakis and M. Matolcsi, Tiles with no spectra, Forum Math., **18** (2006), 519–528.

[22] G. Kutyniok, Beurling density and shift-invariant weighted Gabor systems, *Sampl. Theory Signal Image Process.*, **5** (2006), 163–181.

[23] H. Landau, Sampling, data transmission, and the Nyquist rate, *Proc. IEEE*, **55** (1967), 1701–1706.

[24] H. Landau, Necessary density conditions for sampling and interpolation of certain entire functions, *Acta Math.*, **117** (1967), 37–52.

[25] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, I*, Springer–Verlag, New York, 1977.

[26] M. Matolcsi, Fuglede's conjecture fails in dimension 4, preprint (2005). *Proc. Amer. Math. Soc.*, **133** (2005), 3021–3026.

[27] T. E. Olson and R. A. Zalik, Nonexistence of a Riesz basis of translates, in: *Approximation Theory*, Lecture Notes in Pure and Applied Math., Vol. 138, Dekker, New York, 1992, pp. 401–408.

[28] J. Ramanathan and T. Steger, Incompleteness of sparse coherent states, *Appl. Comput. Harmon. Anal.*, **2** (1995), 148–153.

[29] T. Strohmer and R. W. Heath, Jr., Grassmannian frames with applications to coding and communication, *Appl. Comput. Harmon. Anal.*, **14** (2003), 257–275.

[30] I. Singer, *Bases in Banach Spaces* I, Springer–Verlag, New York, 1970.

[31] T. Tao, Fuglede's conjecture is false in 5 and higher dimensions, *Math. Res. Lett.*, **11** (2004), 251–258.

[32] B. Rom and D. Walnut, Sampling on unions of shifted lattices in one dimension, in: *Harmonic Analysis and Applications*, Birkhäuser, Boston, 2006, pp. 289–323.

[33] R. Young, *An Introduction to Nonharmonic Fourier Series*, Revised First Edition, Academic Press, San Diego, 2001.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332 USA
E-mail address: heil@math.gatech.edu

Justus-Liebig-University Giessen, Institute of Mathematics, 35392 Giessen, Germany
E-mail address: gitta.kutyniok@math.uni-giessen.de