A WEAK QUALITATIVE UNCERTAINTY PRINCIPLE FOR COMPACT GROUPS

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Abstract. For locally compact abelian groups it is known that if the product of the measures of the support of an L^1-function f and its Fourier transform is less than 1 then $f = 0$ almost everywhere. This is a weak version of the classical qualitative uncertainty principle. In this paper we focus on compact groups G and investigate the problem of when there exists a lower bound for all products of the measures of the support of an integrable function and its Fourier transform and when this bound equals 1. These properties are related to the structure of the group G. Finally, exact values which the product can attain are given for several types of compact groups.

1. Introduction

Let G be a separable unimodular locally compact group of type I equipped with left Haar measure m_G. $\widehat{G}$ will denote the dual space of G, i.e. the set of all equivalence classes of irreducible unitary representations. Let μ_G be the Plancherel measure on $\widehat{G}$. For $\pi \in \widehat{G}$, the associated representation space is $\mathcal{H}_\pi$ with dimension d_π. The Fourier transform $\hat{f}$ of a function $f \in L^1(G)$ is defined by

$$\langle \hat{f}(\pi)\xi, \eta\rangle = \int_G f(x)\langle \pi(x^{-1})\xi, \eta\rangle dm_G(x),$$

where $\pi \in \widehat{G}$, $\xi, \eta \in \mathcal{H}_\pi$ and $\langle\cdot,\cdot\rangle$ the inner product on $\mathcal{H}_\pi$. For $f \in L^1(G)$, let $A_f = \{x \in G : f(x) \neq 0\}$ and $B_f = \{\pi \in \widehat{G} : \hat{f}(\pi) \neq 0\}$.

In this paper we will study qualitative uncertainty principles for compact groups. Generally speaking an uncertainty principle shows that a nonzero function and its Fourier transform cannot both be sharply localized. There exists an abundance of special types of uncertainty principles. For an excellent survey we refer to [5]. We speak of a qualitative uncertainty principle if without giving quantitative estimates it is shown that a function and its Fourier transform cannot be too localized unless the function equals zero. The first

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qualitative uncertainty principle of the type we want to discuss here was derived in 1973 by Matolcsi and Szücs [11]. It states the following. Given a locally compact abelian group G, for $f \in L^2(G)$, we have

$$m_G(A_f)\mu_G(B_f) < 1 \quad \Longrightarrow \quad f = 0 \quad \text{a.e..}$$

This result was proven for L^1-functions by Smith [15]. On $\mathbb{R}^n$ a much stronger result is true. In 1985 Benedicks [1] proved that for $f \in L^1(\mathbb{R}^n)$,

$$m_{\mathbb{R}^n}(A_f) < \infty \quad \text{and} \quad \mu_{\mathbb{R}^n}(B_f) < \infty \quad \Longrightarrow \quad f = 0 \quad \text{a.e..}$$

A formulation of the qualitative uncertainty principle which seems to be the right setting for a large class of locally compact groups G and which is referred to as QUP is the following. G is said to satisfy the *QUP* if, for all $f \in L^1(G)$,

$$m_G(A_f) < m_G(G) \quad \text{and} \quad \mu_G(B_f) < \mu_G(\widehat{G}) \quad \Longrightarrow \quad f = 0 \quad \text{a.e..}$$

Hogan [7] proved that the QUP holds for a non-compact non-discrete locally compact abelian group with connected component G_0 if and only if G_0 is non-compact. An infinite compact group satisfies the QUP if and only if it is connected (see Hogan [8]). There exists an abundance of generalizations, e.g. [2, 13, 3, 9, 14].

It seems natural to ask, whether there exists a weaker version of the QUP, which is less restrictive. For this, we consider the principle stated by Matolcsi and Szücs [11], which can be formulated for all separable unimodular locally compact groups G of type I. We say that such a G satisfies the *weak QUP* if, for each $f \in L^1(G)$,

$$m_G(A_f)\mu_G(B_f) < 1 \quad \Longrightarrow \quad f = 0 \quad \text{a.e..}$$

The expectation is that this is satisfied by many more groups than the QUP. Indeed, each locally compact abelian group satisfies the weak QUP even though it may not satisfy the QUP (compare [11], [7] and [8]). In this paper we will focus on compact groups and study the weak QUP and related properties.

The purpose of the second section of this paper is to state some basic results, which will be needed in the following.

Let G be a compact group. In Section 3 we then characterize exactly the weak QUP in terms of the group structure of G (Theorem 1). If G does not satisfy the weak QUP, it is an interesting question whether there still exists a lower bound for the product of the measures of the support of an integrable function and its Fourier transform. We give a sufficient condition for the existence of such a lower bound (Theorem 2). We even obtain an explicit bound. Moreover, the necessity of this condition is proven under some hypothesis on the structure of G. We further give a class of compact groups which satisfy this hypothesis (Proposition 3.1).

Moreover, it is desirable to know which values can be attained by the product $m_G(A_f)\mu_G(B_f)$, where G is a compact group and $f \in L^1(G)$. This would help us to keep the time-frequency localization of the function under control. This question will be investigated in Section 4. In Subsection 4.1 we study whether the lower bounds for $m_G(A_f)\mu_G(B_f)$ obtained in the two theorems are sharp, whereas in Subsection 4.2 we give possible values which are attained by this product for several types of compact groups.

2. Basic results

Let G be a compact group. We will always normalize m_G so that $m_G(G) = 1$. The Plancherel measure μ_G, which is the unique measure on $\widehat{G}$ such that for any $f \in L^1(G) \cap L^2(G)$

$$\int_G |f(x)|^2 dm_G(x) = \int_{\widehat{G}} \mathrm{tr}[\hat{f}(\pi)^* \hat{f}(\pi)] d\mu_G(\pi),$$

is then given by

$$\mu_G(F) = \sum_{\pi \in F} d_\pi \quad \text{for every subset } F \subseteq \widehat{G}.$$

Here $\mathrm{tr}[\cdot]$ denotes the trace of an operator. Let $1_{\mathcal{H}_\pi}$ be the identity operator on a Hilbert space $\mathcal{H}_\pi$ and χ_E the characteristic function of a measurable subset E of G. If M is a finite set, the number of elements shall be denoted by $|M|$. Let G_0 denote the connected component of the identity in G. The annihilator of a closed subgroup H of G in $\widehat{G}$ is defined by

$$A(H, \widehat{G}) = \{\pi \in \widehat{G} : \pi(h) = 1_{\mathcal{H}_\pi} \text{ for all } h \in H\}.$$

If H is a closed normal subgroup, $A(H, \widehat{G})$ can be identified with $\widehat{G/H}$ [6, Corollary 28.10]. For more information on Fourier analysis on compact groups we refer to Folland [4].

In the sequel we will be often dealing with functions $f \in L^1(G)$ which are constant on cosets of some closed normal subgroup. For determining B_f we need to know its Fourier transform. The following lemma is folklore, but since we couldn't find a reference, a short proof is included for completeness.

Lemma 2.1. *Let G be a compact group, let H be a closed normal subgroup of G and let $\varphi : G \to G/H$ denote the quotient map. Further, let $f \in L^1(G)$ be such that there exists some function $g \in L^1(G/H)$ with $f(x) = g(\varphi(x))$. Then, for $\pi \in \widehat{G}$ and $\xi, \eta \in \mathcal{H}_\pi$, we have*

$$\langle \hat{f}(\pi)\xi, \eta \rangle = \chi_{A(H,\widehat{G})}(\pi) \langle \hat{g}(\pi)\xi, \eta \rangle.$$

Proof. Using Weil's formula, the Schur orthogonality relations and the fact that unitary representations of compact groups are direct sums of irreducible ones [4, Theorem 5.2], we obtain

$$\langle \hat{f}(\pi)\xi, \eta\rangle = \int_{G/H} g(yH)\chi_{A(H,\widehat{G})}(\pi)\langle \pi(y^{-1})\xi, \eta\rangle dm_{G/H}(yH).$$

If $\pi \notin A(H,\widehat{G})$, we have $\hat{f}(\pi) = 0$. Now let $\pi \in A(H,\widehat{G})$. Then we obtain

$$\langle \hat{f}(\pi)\xi, \eta\rangle = \langle \hat{g}(\pi)\xi, \eta\rangle.$$

□

The next two lemmas will be used throughout the proof of Theorems 1 and 2.

Lemma 2.2. *Let G be a compact Lie group and let $f \in L^1(G)$, $f \neq 0$. Then there exists a function g on G/G_0, $g \neq 0$ such that*

$$m_G(A_f)\mu_G(B_f) \geq m_{G/G_0}(A_g)\mu_{G/G_0}(B_g).$$

Proof. Let $f \in L^1(G)$, $f \neq 0$ and let $\{x_i : i = 1, \ldots, [G : G_0]\}$ be a representative system for the G_0-cosets in G. We define g on G/G_0 by $g(x_i) = \int_{G_0} f(x_i h)dm_{G_0}(h)$ and $k \in L^1(G)$ by $k(x) = g(\varphi(x))$, where $\varphi : G \to G/G_0$ is the quotient map. Without loss of generality we can assume that $\mu_G(B_f) < \infty$. This means precisely that f equals a trigonometric polynomial almost everywhere. Since G is also a Lie group, such an f must be analytic. Let $x \in G$ and consider $f|_{xG_0}$. This is also an analytic function which is defined on a connected set. But nonzero analytic functions, defined on a connected set, cannot vanish on a set of positive measure. This shows that for each $x \in G$, we have either $f|_{xG_0} \neq 0$ a.e. or $f|_{xG_0} \equiv 0$. Thus, by definition of the function k, $A_k \subseteq A_f$ and hence $m_G(A_f) \geq m_G(A_k)$. The normalization of the measures m_G and m_{G/G_0} implies that $m_G(A_k) = m_{G/G_0}(A_g)$.

Now we will prove $\mu_G(B_f) \geq \mu_{G/G_0}(B_g)$. Using Weil's formula, for each $\pi \in \widehat{G}$ and $\xi, \eta \in \mathcal{H}_\pi$, we obtain

$$\begin{aligned}
&\langle \hat{f}(\pi)\xi, \eta\rangle \\
&= \int_G f(x)\langle \pi(x^{-1})\xi, \eta\rangle dm_G(x) \\
&= \frac{1}{[G:G_0]}\sum_{i=1}^{[G:G_0]}\int_{G_0} f(x_i h)\langle \pi(h^{-1})\pi(x_i^{-1})\xi, \eta\rangle dm_{G_0}(h) \\
&= \frac{1}{[G:G_0]}\begin{cases} \sum_{i=1}^{[G:G_0]} \int_{G_0} f(x_i h)dm_{G_0}(h)\langle \pi(x_i^{-1})\xi, \eta\rangle & : \pi \in A(G_0,\widehat{G}), \\ \sum_{i=1}^{[G:G_0]} \langle \widehat{f(x_i\,\cdot)}(\pi)(\pi(x_i^{-1})\xi), \eta\rangle & : \pi \notin A(G_0,\widehat{G}). \end{cases}
\end{aligned}$$

This shows $\langle \hat{f}(\pi)\xi, \eta\rangle = \langle \hat{k}(\pi)\xi, \eta\rangle$ for all $\pi \in A(G_0, \widehat{G})$ and $\xi, \eta \in \mathcal{H}_\pi$. Applying Lemma 2.1 yields that, for each $\pi \in \widehat{G}$ and $\xi, \eta \in \mathcal{H}_\pi$,

$$\langle \hat{k}(\pi)\xi, \eta\rangle = \chi_{A(G_0,\widehat{G})}(\pi)\langle \hat{g}(\pi)\xi, \eta\rangle.$$

Thus, by the structure of the Plancherel measure, we obtain $\mu_G(B_f) \geq \mu_{G/G_0}(B_g)$. □

Lemma 2.3. *Let G be a compact group and let $f \in L^1(G)$, $f \neq 0$. Then there exist a closed normal subgroup H of G such that G/H is Lie and a function $g \in L^1(G/H)$ such that*

$$m_G(A_f)\mu_G(B_f) = m_{G/H}(A_g)\mu_{G/H}(B_g).$$

Proof. Each compact group is a projective limit of Lie groups (see [6, 28.61 (c)]), i.e. there exists a system $\mathcal{L}$ of closed normal subgroups H of G, $\mathcal{L}$ downwards directed and $\bigcap_{H\in\mathcal{L}} H = \{e\}$, such that G/H is a compact Lie group for every $H \in \mathcal{L}$. Moreover, $\widehat{G}$ is the corresponding injective limit of the annihilators $A(H, \widehat{G})$, $H \in \mathcal{L}$. Let $f \in L^1(G)$, $f \neq 0$ with $\mu_G(B_f) < \infty$. By the Fourier inversion formula, f can be represented in the following way

$$f(x) = \sum_{i=1}^{n} d_{\pi_i} \mathrm{tr}[\hat{f}(\pi_i)\pi_i(x)].$$

Now there exists some $H \in \mathcal{L}$ such that $\pi_i \in A(H, \widehat{G})$ for all $1 \leq i \leq n$. For $h \in H$, $f(xh) = f(x)$ since $\pi_i(h) = 1_{\mathcal{H}_{\pi_i}}$. Let $\varphi : G \to G/H$ be the canonical quotient map and define $g \in L^1(G/H)$ by $g(\varphi(x)) = f(x)$. Then $m_G(A_f) = m_{G/H}(A_g)$, since m_G and $m_{G/H}$ are chosen to be normalized.

To prove $\mu_G(B_f) = \mu_{G/H}(B_g)$, let $\pi \in \widehat{G}$ and let $\xi, \eta \in \mathcal{H}_\pi$. Lemma 2.1 implies that

$$\langle \hat{f}(\pi)\xi, \eta\rangle = \chi_{A(H,\widehat{G})}(\pi)\langle \hat{g}(\pi)\xi, \eta\rangle.$$

Employing now the structure of the Plancherel measure yields $\mu_G(B_f) = \mu_{G/H}(B_g)$. □

3. The weak QUP and related properties

Let G be a compact group. We first characterize the weak QUP in terms of the group structure of G. We obtain an equivalent condition for the weak QUP to hold which is satisfied by a larger set of compact groups than just the connected ones. This shows that indeed the weak QUP is much less restrictive than the QUP.

Theorem 1. *Let G be a compact group. The following conditions are equivalent.*

(i) *G satisfies the weak QUP.*
(ii) *G/G_0 is abelian.*

Proof. Let G be a compact group. Towards a contradiction we assume that G/G_0 is non-abelian. Since G/G_0 is also totally disconnected, there exists an open normal subgroup C of G/G_0 such that $(G/G_0)/C$ is non-abelian. Let H be the pullback of C to G. Then G/H is finite and non-abelian. We define $f \in L^1(G)$ by $f = \chi_H$. Then, since $m_G(G) = 1$, we have $m_G(A_f) = [G : H]^{-1}$. In order to calculate $\mu_G(B_f)$, let $\pi \in \widehat{G}$ and $\xi, \eta \in \mathcal{H}_\pi$. Then, by Lemma 2.1,

$$\langle \hat{f}(\pi)\xi, \eta\rangle = \frac{1}{[G:H]}\chi_{A(H,\widehat{G})}(\pi)\langle \xi, \eta\rangle.$$

Let $A(H, \widehat{G})$ be identified with $\widehat{G/H}$. Now the definition of the Plancherel measure implies $\mu_G(B_f) = \sum_{\pi\in\widehat{G/H}} d_\pi$. Since G/H is non-abelian, there exists at least one $\pi \in \widehat{G/H}$ with $d_\pi > 1$. Thus $\sum_{\pi\in\widehat{G/H}} d_\pi < \sum_{\pi\in\widehat{G/H}} d_\pi^2$. Since G/H is a finite group, we have $[G : H] = \sum_{\pi\in\widehat{G/H}} d_\pi^2$ (see [4, Proposition 5.27]). This shows $\mu_G(B_f) < [G : H]$, which in turn implies $m_G(A_f)\mu_G(B_f) < 1$. This proves (i) $\Rightarrow$ (ii).

Now suppose (ii) holds. Our purpose is to show that then G satisfies the weak QUP. This will be achieved by first reducing to compact Lie groups and then to finite groups. Let G be an arbitrary compact group and let $f \in L^1(G)$, $f \neq 0$. Lemma 2.3 implies that there exist a closed normal subgroup H such that G/H is Lie and a function $g \in L^1(G/H)$ such that

$$m_G(A_f)\mu_G(B_f) = m_{G/H}(A_g)\mu_{G/H}(B_g).$$

Note that $G/G_0H = (G/H)/(G_0H/H)$ and, since G_0H/H is connected and open in G/H, we have $G_0H/H = (G/H)_0$. By hypothesis, G/G_0 is abelian. Thus also $(G/H)/(G/H)_0$ is abelian. Hence we can assume that G is a compact Lie group. In this situation, we may apply Lemma 2.2, which shows the existence of some function $g \in L^1(G/G_0)$, $g \neq 0$ such that

$$m_G(A_f)\mu_G(B_f) \geq m_{G/G_0}(A_g)\mu_{G/G_0}(B_g).$$

Since G/G_0 is supposed to be abelian, applying [11] yields

$$m_{G/G_0}(A_g)\mu_{G/G_0}(B_g) \geq 1.$$

This finishes the proof. □

Let G be some compact group which does not satisfy the weak QUP. The following theorem studies necessary and sufficient conditions for the existence of a lower bound for $m_G(A_f)\mu_G(B_f)$ for all $f \in L^1(G)$, $f \neq 0$. For this, we define $\mathcal{H}$ to be the set of all compact open normal subgroups H of G. Recall that an open subgroup of a locally compact group G always contains G_0.

A locally compact group G is called *almost abelian* if it contains an abelian normal subgroup of finite index. Moore [12] proved that for an arbitrary locally compact group G, the existence of an abelian normal subgroup of finite index is equivalent to $\max_{\pi\in\widehat{G}} d_\pi < \infty$.

Theorem 2. *Let G be a compact group. Consider the following conditions.*

(i) *There exists $M > 0$ such that $m_G(A_f)\mu_G(B_f) \geq M$ for all $f \in L^1(G)$, $f \neq 0$.*
(ii) *G/G_0 is almost abelian.*

Then (ii) *implies* (i), *and M can be chosen as $(\max_{\pi\in\widehat{G/G_0}} d_\pi)^{-1}$.*

Conversely, if

$$\inf_{H\in\mathcal{H}} \frac{\sum_{\pi\in\widehat{G/H}} d_\pi}{\sum_{\pi\in\widehat{G/H}} d_\pi^2} = 0$$

under the assumption that G/G_0 is not almost abelian, then (i) *implies* (ii).

Proof. Let G be a compact group. First suppose that G/G_0 is almost abelian. Let $f \in L^1(G)$, $f \neq 0$. By Lemma 2.2 and Lemma 2.3, there exist a closed normal subgroup H of G such that G/H is Lie and a function g on $(G/H)/(G/H)_0 = G/G_0H$ such that

$$m_G(A_f)\mu_G(B_f) \geq m_{G/G_0H}(A_g)\mu_{G/G_0H}(B_g).$$

Moreover, we have

$$\max_{\pi\in\widehat{G/G_0H}} d_\pi \leq \max_{\pi\in\widehat{G/G_0}} d_\pi < \infty.$$

For the last inequality see [12, Proposition 2.1].

Now let G be a finite group. By the preceding paragraph and since G/G_0H is finite, it suffices to prove $m_G(A_f)\mu_G(B_f) \geq (\max_{\pi\in\widehat{G}} d_\pi)^{-1}$ for each function f on G, $f \neq 0$. For this, let f be some function on G, $f \neq 0$. For each $\pi \in \widehat{G}$, we may identify $\mathcal{H}_\pi$ with $\mathbb{C}^{d_\pi}$ and denote its canonical orthonormal basis by $\{\xi_i : i = 1, \ldots, d_\pi\}$. Then $\pi(x)$, where $x \in G$, can be represented by a matrix with respect to this basis, which we will denote by $(\pi_{ij}(x))_{1\leq i,j\leq d_\pi}$. First we get

$$\begin{aligned}
\mathrm{tr}[\hat{f}(\pi)^*\hat{f}(\pi)] &= \sum_{i=1}^{d_\pi} \langle \hat{f}(\pi)\xi_i, \hat{f}(\pi)\xi_i\rangle \\
&= \frac{1}{|G|^2}\sum_{i=1}^{d_\pi}\sum_{x,y\in G} f(x)\overline{f(y)}\,\pi_{ii}(yx^{-1}) \\
&\leq \frac{1}{|G|^2}\sum_{i=1}^{d_\pi}\sum_{x,y\in G} |f(x)||f(y)||\pi_{ii}(yx^{-1})| \\
&\leq \frac{1}{|G|^2}\sum_{i=1}^{d_\pi}\sum_{x,y\in G} |f(x)||f(y)|.
\end{aligned}$$

Using the Plancherel formula, this inequality and Hölder's inequality, we obtain

$$\|f\|_2^2 \leq \mu_G(B_f) \max_{\pi\in\widehat{G}} \operatorname{tr}[\hat{f}(\pi)^*\hat{f}(\pi)] \tag{1}$$

$$\leq \mu_G(B_f)(\max_{\pi\in\widehat{G}} d_\pi)\|f\|_1^2 \tag{2}$$

$$\leq \mu_G(B_f) m_G(A_f)(\max_{\pi\in\widehat{G}} d_\pi)\|f\|_2^2. \tag{3}$$

This shows

$$m_G(A_f)\mu_G(B_f) \geq \frac{1}{\max_{\pi\in\widehat{G}} d_\pi}.$$

Now suppose that G/G_0 is not almost abelian. Let $H \in \mathcal{H}$. We define $f_H \in L^1(G)$ by $f_H = \chi_H$. Our choice of Haar measures on compact groups implies $m_G(A_{f_H}) = [G : H]^{-1}$. Concerning the Fourier transform of f_H, Lemma 2.1 shows that, for each $\pi \in \widehat{G}$ and $\xi, \eta \in \mathcal{H}_\pi$,

$$\langle \widehat{f_H}(\pi)\xi, \eta\rangle = \frac{1}{[G : H]}\chi_{A(H,\widehat{G})}(\pi)\langle \xi, \eta\rangle.$$

We identify $A(H, \widehat{G})$ with $\widehat{G/H}$. Then the definition of the Plancherel measure implies $\mu_G(B_{f_H}) = \sum_{\pi\in\widehat{G/H}} d_\pi$. Hence, using [4, Proposition 5.27], we get

$$m_G(A_{f_H})\mu_G(B_{f_H}) = \frac{1}{[G : H]} \sum_{\pi\in\widehat{G/H}} d_\pi = \frac{\sum_{\pi\in\widehat{G/H}} d_\pi}{\sum_{\pi\in\widehat{G/H}} d_\pi^2}.$$

By hypothesis, we have

$$\inf_{H\in\mathcal{H}} \frac{\sum_{\pi\in\widehat{G/H}} d_\pi}{\sum_{\pi\in\widehat{G/H}} d_\pi^2} = 0,$$

which implies

$$\inf_{H\in\mathcal{H}} m_G(A_{f_H})\mu_G(B_{f_H}) = 0.$$

□

The next result gives an explicit class of compact groups for which we have equivalence of condition (i) and (ii).

Proposition 3.1. *Let G be a compact group such that G/G_0 is a direct product of finite groups. Then the conditions* (i) *and* (ii) *of Theorem 2 are equivalent.*

Proof. Let G be a compact group such that G/G_0 is a direct product of finite groups. Suppose that G/G_0 is not almost abelian. This implies that there exist an abelian group A and infinitely many finite non-abelian groups F_j, $j \in \mathbb{N}$ with $G/G_0 = A \times \prod_{j=1}^{\infty} F_j$. By Theorem 2, it suffices to prove that

$$\inf_{H \in \mathcal{H}} \frac{\sum_{\pi \in \widehat{G/H}} d_\pi}{\sum_{\pi \in \widehat{G/H}} d_\pi^2} = 0.$$

Let H_n, $n \in \mathbb{N}$ be those subgroups of G which satisfy $H_n/G_0 = A \times \prod_{j=n+1}^{\infty} F_j$, where we regard the direct product as a subgroup of G/G_0 in the canonical way. Then, for each $n \in \mathbb{N}$, we have $H_n \in \mathcal{H}$. Moreover, we define G_n by $G_n = G/H_n = (G/G_0)/(H_n/G_0) = \prod_{j=1}^{n} F_j$. For simplicity, we set

$$q(J) = |J|^{-1} \sum_{\pi \in \widehat{J}} d_\pi$$

for any finite group J.

We claim that

$$q(G_n) \to 0 \quad \text{as } n \to \infty.$$

For this, we first note that $J = B \times C$ implies $q(J) = q(B)q(C)$, since $\widehat{J} = \widehat{B} \times \widehat{C}$. Let $k = |J'|$, where J' denotes the commutator subgroup of J. Then

$$\begin{aligned} q(J) &= |J|^{-1} \left(|J/J'| + \sum_{\pi \in \widehat{J},\, d_\pi \geq 2} d_\pi \right) \\ &= \frac{1}{k} + |J|^{-1} \sum_{\pi \in \widehat{J},\, d_\pi \geq 2} d_\pi \\ &\leq \frac{1}{k} + \frac{1}{2}|J|^{-1} \sum_{\pi \in \widehat{J},\, d_\pi \geq 2} d_\pi^2 \\ &< \frac{1}{k} + \frac{1}{2}. \end{aligned}$$

Let $n \in \mathbb{N}$. Since F_{n+1} and F_{n+2} are both non-abelian, their commutator subgroups have order at least 2, so the commutator subgroup of $F_{n+1} \times F_{n+2}$ has order at least 4, whence $q(F_{n+1} \times F_{n+2}) < \frac{3}{4}$ by the preceding calculation. Therefore we obtain

$$q(G_{n+2}) = q(G_n)q(F_{n+1} \times F_{n+2}) < \frac{3}{4} q(G_n).$$

This proves the claim.

Finally, we have

$$\inf_{H \in \mathcal{H}} \frac{\sum_{\pi \in \widehat{G/H}} d_\pi}{\sum_{\pi \in \widehat{G/H}} d_\pi^2} \leq \inf_{n \in \mathbb{N}} q(G_n) = 0.$$

□

Theorem 2 and Proposition 3.1 lead to the following conjecture.

Conjecture. *Let G be a compact group. Then the conditions* (i) *and* (ii) *of Theorem 2 are equivalent.*

4. Values of $m_G(A_f)\mu_G(B_f)$

After determining conditions which guarantee the weak QUP to hold and which guarantee the existence of a lower bound for $m_G(A_f)\mu_G(B_f)$, we are now interested in possible values of this product.

4.1. **Lower bounds.** Let G be a compact group and let $f \in L^1(G)$, $f \neq 0$. In this subsection we will study lower bounds for the product $m_G(A_f)\mu_G(B_f)$.

First we consider the situation when G/G_0 is abelian. By Theorem 1, the value 1 is a lower bound. It is easy to show that this bound is always sharp. Let $f \in L^1(G)$ be defined by $f = \chi_G$. Then f satisfies

$$m_G(A_f)\mu_G(B_f) = 1.$$

Obviously, any function $f_H \in L^1(G)$ defined by $f_H = \chi_H$, where H is a compact open normal subgroup of G, fulfills this equation. It is interesting to note that provided G is an infinite compact group which does not satisfy the QUP, for some closed normal subgroup H of G the function f_H not only attains the infimum but even violates the QUP, i.e. $m_G(A_f) < m_G(G)$ and $\mu_G(B_f) < \mu_G(\widehat{G})$. We just have to choose some proper open compact normal subgroup H of G which is non-trivial. Such a subgroup exists, since the hypothesis implies that G is not connected (see [8, Theorem 2.6]) and hence G/G_0 is a non-trivial totally disconnected compact group. Now we can apply [6, Theorem 7.7].

Let us mention that in the situation of locally compact abelian groups, we can exactly classify all functions $f \in L^2(G)$ for which $m_G(A_f)\mu_G(B_f)$ attains the infimum, i.e. $m_G(A_f)\mu_G(B_f) = 1$ (see [10, Theorem 2.4]).

Secondly, we examine the situation when G/G_0 is almost abelian. Theorem 2 shows that $(\max_{\pi \in \widehat{G/G_0}} d_\pi)^{-1}$ is a lower bound. Again the question arises whether this bound is also sharp. We can easily construct functions satisfying

$$m_G(A_f)\mu_G(B_f) = \frac{\sum_{\pi \in \widehat{G/H}} d_\pi}{\sum_{\pi \in \widehat{G/H}} d_\pi^2},$$

where H is an open compact normal subgroup of G, by setting $f = \chi_H$. However, we can show that the bound $(\max_{\pi \in \widehat{G/G_0}} d_\pi)^{-1}$ can never be attained.

Proposition 4.1. *Let G be a compact group such that G/G_0 is almost abelian, but not abelian. Then, for each $f \in L^1(G)$, $f \neq 0$ we have*

$$m_G(A_f)\mu_G(B_f) > \frac{1}{\max_{\pi \in \widehat{G/G_0}} d_\pi}.$$

Proof. Let $f \in L^1(G)$, $f \neq 0$. Lemma 2.2 and Lemma 2.3 imply that we can find some function g on $(G/H)/(G/H)_0 = G/G_0H$, $g \neq 0$ such that

$$m_G(A_f)\mu_G(B_f) \geq m_{G/G_0H}(A_g)\mu_{G/G_0H}(B_g),$$

where H is a closed normal subgroup of G such that G/H is Lie. Without loss of generality we can assume that G/G_0H is non-abelian. Moreover, we have

$$\max_{\pi \in \widehat{G/G_0H}} d_\pi \leq \max_{\pi \in \widehat{G/G_0}} d_\pi < \infty.$$

Now let G be a finite non-abelian group. By the preceding paragraph it suffices to prove $m_G(A_f)\mu_G(B_f) > (\max_{\pi \in \widehat{G}} d_\pi)^{-1}$ for all functions f on G, $f \neq 0$. Towards a contradiction assume that there exists some function f on G which satisfies

$$m_G(A_f)\mu_G(B_f) = \frac{1}{\max_{\pi \in \widehat{G}} d_\pi}.$$

Throughout the proof, for each $\pi \in \widehat{G}$, we identify $\mathcal{H}_\pi$ with $\mathbb{C}^{d_\pi}$ and denote its standard orthonormal basis by $\{\xi_i : i = 1, \ldots, d_\pi\}$. Furthermore, for $x \in G$, let the matrix of $\pi(x)$, $(\pi_{ij}(x))_{1 \leq i,j \leq d_\pi}$, be chosen with respect to this basis.

The assumption implies that we must have equality in (1) to (3). This is fulfilled if and only if there exist $c, d > 0$ such that

(i) $\sum_{i=1}^{d_\pi} \langle \hat{f}(\pi)\xi_i, \hat{f}(\pi)\xi_i \rangle = d$ for all $\pi \in B_f$,
(ii) $d = (\max_{\rho \in \widehat{G}} d_\rho) m_G(A_f)^2 c^2$,
(iii) $|\pi_{ii}(yx^{-1})| = 1$ for all $x, y \in A_f$, $\pi \in B_f$ and $1 \leq i \leq d_\pi$,
(iv) $|f(x)| = c\chi_{A_f}(x)$ for all $x \in G$.

In detail, (i) is equivalent to equality in (1), (iv) is equivalent to equality in (3) and then (ii) and (iii) hold if and only if we have equality in (2). Without loss of generality we can assume that $c = 1$.

Let $x, y \in A_f$. Using the Cauchy-Schwarz inequality, it follows from (iii) that ξ_i is an eigenvector of $\pi(yx^{-1})$ for all $1 \leq i \leq d_\pi$. By choice of the basis $\{\xi_i : i = 1, \ldots, d_\pi\}$, this in turn implies that the matrix $(\pi_{ij}(yx^{-1}))_{1 \leq i,j \leq d_\pi}$ is diagonal. Without loss of generality we can assume that $e \in A_f$, since otherwise we could choose some $x_0 \in A_f$ and consider the function $g := f(x_0 \cdot)$. Then we would have $e \in A_g$, $m_G(A_g) = m_G(A_f)$ and $\mu_G(B_g) = \mu_G(B_f)$, because $\hat{g}(\pi) = \hat{f}(\pi)\pi(x_0)$, $\pi \in \widehat{G}$. Thus also $(\pi_{ij}(y))_{1 \leq i,j \leq d_\pi}$ is diagonal. In addition, this implies $\pi_{ii}(yx^{-1}) = \pi_{ii}(y)\pi_{ii}(x^{-1})$ for all $1 \leq i \leq d_\pi$.

By conditions (ii), (iii) and (iv), for all $\pi \in B_f$, we have

$$\frac{1}{|G|^2} \sum_{i=1}^{d_\pi} \left| \sum_{x,y \in G} f(x)\overline{f(y)}\, \pi_{ii}(yx^{-1}) \right| = \sum_{i=1}^{d_\pi} \langle \hat{f}(\pi)\xi_i, \hat{f}(\pi)\xi_i \rangle$$

$$= (\max_{\rho \in \widehat{G}} d_\rho) m_G(A_f)^2 = (\max_{\rho \in \widehat{G}} d_\rho) \frac{1}{|G|^2} \sum_{x,y \in G} |f(x)||f(y)||\pi_{ii}(yx^{-1})|.$$

This implies immediately

$$d_\pi = \max_{\rho \in \widehat{G}} d_\rho \quad \text{for all } \pi \in B_f \tag{4}$$

and

$$\left| \sum_{x,y \in G} f(x)\overline{f(y)}\, \pi_{ii}(yx^{-1}) \right| = \sum_{x,y \in G} |f(x)||f(y)||\pi_{ii}(yx^{-1})|.$$

Thus there exists some constant $\lambda_{\pi_{ii}}$ [6, Theorem 12.4] such that

$$f(x)\overline{f(y)}\, \pi_{ii}(yx^{-1}) = f(x)\overline{f(y)}\, \pi_{ii}(y)\pi_{ii}(x^{-1}) = \lambda_{\pi_{ii}}$$

for all $x, y \in A_f$, $\pi \in B_f$ and $1 \le i \le d_\pi$. If we choose $x = y$ and use (iv), we obtain $\lambda_{\pi_{ii}} = 1$ for all $\pi \in B_f$, $1 \le i \le d_\pi$. This implies the existence of a λ with $|\lambda| = 1$ and

$$\overline{f(y)}\pi_{ii}(y) = \lambda \quad \text{for all } y \in A_f, \pi \in B_f \text{ and } 1 \le i \le d_\pi.$$

Let $(\hat{f}(\pi)_{ij})_{1 \le i,j \le d_\pi}$ denote the matrix of $\hat{f}(\pi)$ with respect to the basis $\{\xi_i : i = 1, \ldots, d_\pi\}$. Then, for each $\pi \in B_f$,

$$\hat{f}(\pi)_{ii} = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{\pi_{ii}(x)} = \overline{\lambda} m_G(A_f) \quad \text{for all } 1 \le i \le d_\pi.$$

Since $(\pi_{ij}(x))_{1 \le i,j \le d_\pi}$ is diagonal, $\hat{f}(\pi)_{ij} = 0$ for all $\pi \in B_f$, $i \ne j$. Next we will calculate f by employing the inverse Fourier transform. For all $x \in G$, we get

$$f(x) = \sum_{\pi \in \widehat{G}} d_\pi \sum_{i=1}^{d_\pi} \hat{f}(\pi)_{ii}\pi_{ii}(x) = \overline{\lambda} m_G(A_f)(\max_{\rho \in \widehat{G}} d_\rho) \sum_{\pi \in B_f} \sum_{i=1}^{d_\pi} \pi_{ii}(x). \tag{5}$$

Here we also used (4). Now let $x \in A_f$ and $\pi \in B_f$. Since, by assumption, the function f satisfies $m_G(A_f)\mu_G(B_f) = (\max_{\rho \in \widehat{G}} d_\rho)^{-1}$ and (iv), we obtain

$$\frac{1}{\mu_G(B_f)} \left| \sum_{\pi \in B_f} \sum_{i=1}^{d_\pi} \pi_{ii}(x) \right| = m_G(A_f)(\max_{\rho \in \widehat{G}} d_\rho) \left| \sum_{\pi \in B_f} \sum_{i=1}^{d_\pi} \pi_{ii}(x) \right| = |f(x)| = 1$$

which in turn implies

$$\left| \sum_{\pi \in B_f} \sum_{i=1}^{d_\pi} \pi_{ii}(x) \right| = \mu_G(B_f).$$

However, since $\mu_G(B_f) = |B_f|(\max_{\rho \in \widehat{G}} d_\rho)$, this equality can only be fulfilled if $\pi_{ii}(x) = 1$ for all $1 \le i \le d_\pi$.

Next we will show that A_f equals a normal subgroup of G. For this, let $x \notin A_f$. Then, by (5), there exists some $\pi \in B_f$ with $\pi(x) \ne 1_{\mathcal{H}_\pi}$. On the other hand, we just proved that for all $x \in A_f$, we must have $\pi(x) = 1_{\mathcal{H}_\pi}$ for all $\pi \in B_f$. Thus

$$A_f = \{x \in G : \pi(x) = 1_{\mathcal{H}_\pi} \text{ for all } \pi \in B_f\},$$

which is a normal subgroup of G. Now Lemma 2.1 shows that $B_f = A(A_f, \widehat{G})$. However, this implies that B_f contains the trivial representation, which contradicts (4), since G is supposed to be non-abelian. □

4.2. **Values attained.** Let G be a compact group. In this subsection we study the question of which values the product $m_G(A_f)\mu_G(B_f)$, $f \in L^1(G)$ can attain. Notice that the following proofs show how to construct a function $f \in L^1(G)$ to obtain a special value.

Proposition 4.2. *Let G be a compact group. For each $M \subseteq \{\pi \in \widehat{G} : \operatorname{tr}[\pi(x)] \neq 0$ for almost all $x \in G\}$, there exists a function $f \in L^1(G)$ such that*

$$m_G(A_f)\mu_G(B_f) = \sum_{\pi \in M} d_\pi.$$

Proof. Let $M \subseteq \widehat{G}$ be fixed. If $|M| = \infty$, we just have to choose $f \in L^1(G)$ in such a way that $\mu_G(B_f) = \infty$, which trivially exists.

It remains to deal with the case when $|M|$ is finite. For this, let $f \in L^1(G)$ be defined by its Fourier transform

$$\hat{f} = \sum_{\pi \in M} a_\pi \chi_{\{\pi\}} 1_{\mathcal{H}_\pi},$$

where $a_\pi \neq 0$, $\pi \in M$ will be fixed later. Obviously, $\mu_G(B_f) = \sum_{\pi \in M} d_\pi$. Applying the inverse Fourier transform yields

$$f(x) = \sum_{\pi \in M} a_\pi d_\pi \operatorname{tr}[\pi(x)].$$

We have $\operatorname{tr}[\pi(x)] \neq 0$ for almost all $x \in G$. Moreover, G is compact and M is finite. Notice that, if X is a measure space with finite measure and $f, g : X \to \mathbb{C}$ such that $f, g \neq 0$ almost everywhere, then there always exists some $a \in \mathbb{C}$, $a \neq 0$ with $f \neq ag$ almost everywhere. Thus we may choose $a_\pi \neq 0$, $\pi \in M$ in such a way that $f(x) \neq 0$ for almost all $x \in G$. Then f satisfies $m_G(A_f) = 1$, which finishes the proof. □

If, in addition, G is abelian, the previous proposition reduces to the following form.

Corollary 4.3. *Let G be a compact abelian group. For each $n \in \{1, \ldots, |\widehat{G}|\}$, there exists a function $f \in L^1(G)$ such that*

$$m_G(A_f)\mu_G(B_f) = n.$$

Proof. Since G is abelian, $d_\pi = 1$ for all $\pi \in \widehat{G}$. Moreover, $\omega(x) \neq 0$ for all $x \in G$, $\omega \in \widehat{G}$. Hence the claim is an immediate consequence of Proposition 4.2. □

There exist compact groups G for which the product $m_G(A_f)\mu_G(B_f)$, $f \in L^1(G)$ can attain no other values than the ones discussed in Proposition 4.2.

Remark 4.4. Let G be a compact connected group. Then G satisfies the QUP [8, Theorem 2.6]. Hence, for each $f \in L^1(G)$, $f \neq 0$, either $m_G(A_f) = 1$ or $\mu_G(B_f) = \infty$. Thus the numbers $\sum_{\pi \in M} d_\pi$, $M \subseteq \widehat{G}$ are the only possible values, which $m_G(A_f)\mu_G(B_f)$, $f \in L^1(G)$ can attain. In addition, for all $\pi \in \widehat{G}$, we have $\mathrm{tr}[\pi(x)] \neq 0$ for almost all $x \in G$. This follows by standard arguments from the fact that G is connected.

Although Proposition 4.2 includes finite groups, we can obtain a stronger result for finite abelian groups.

Proposition 4.5. *Let G be a finite abelian group. For each $1 \leq p \leq |G|$, $q \in \{|G| - p + 1, \ldots, |G|\}$, there exists a function f on G such that*

$$m_G(A_f)\mu_G(B_f) = \frac{pq}{|G|}.$$

Proof. By the structure theorem, G is of the form $G = \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_s}$ for integers $m_1, \ldots, m_s$ greater than 1, each of which is a power of a prime. We only treat the case $G = \mathbb{Z}_m$. The general case can be proven similarly.

Throughout this proof we identify G with $\widehat{G}$ in the canonical way (compare [6, Example 23.27 (d)]). Let $p \in \{1, \ldots, |G| = m\}$ and $q \in \{m-p+1, \ldots, m\}$ be fixed. We construct a function f on G which satisfies $m_G(A_f) = \frac{q}{m}$ and $\mu_G(B_f) = p$. For this, we have to consider the matrix

$$T := \left(e^{2\pi i \frac{jk}{m}}\right)_{1 \leq j,k \leq m}.$$

If we set $d = e^{2\pi i \frac{1}{m}}$, we can write T in the form

$$T = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & d & d^2 & \ldots & d^{m-1} \\ 1 & d^2 & d^4 & \ldots & d^{2(m-1)} \\ \vdots & & & \ddots & \vdots \\ 1 & d^{m-1} & d^{2(m-1)} & \ldots & d^{(m-1)^2} \end{pmatrix}.$$

Notice that this is a Vandermonde matrix. Let $T_{r,s}$ denote the matrix consisting of the first r rows and s columns of T. Since $m - q < p$ and hence the rank of $T_{m-q,p}$ is $m - q$, the subspace $V_{p,q} = \{a \in \mathbb{C}^p : T_{m-q,p}a = 0\}$ has dimension ≥ 1.

Next we define q' by $q' = m - p + 1$. Then $m - q' = p - 1$. Hence the dimension of $V_{p,q'}$ equals 1. Let $b_0 \in V_{p,q'}$, $b_0 \neq 0$. Towards a contradiction assume that the ith $(1 \leq i \leq p)$ component of b_0 is equal to zero. If the ith column of $T_{m-q',p}$ is deleted, the new matrix is a transpose of a Vandermonde matrix, hence nonsingular. Then all other components of b_0 have to be equal to zero. This is a contradiction. Thus all components of b_0 are nonzero. We define $\tilde{b}_0 \in \mathbb{C}^m$ by $\tilde{b}_0 = (b_0, 0, \dots, 0)^t$. Now we claim that the last $m - p + 1$ components of $T\tilde{b}_0$ are all nonzero. For this, let $i \in \{p, \dots, m\}$ be arbitrarily chosen. We consider the $p \times p$-matrix consisting of $T_{m-q',p}$ and the first p components of the ith row of T as last row. This is again a Vandermonde matrix, hence nonsingular. Thus the ith component of $T\tilde{b}_0$ must not equal zero. This proves the assertion.

Now let $b \in V_{p,q}$ be defined such that the last $p - (m - q)$ components of $T_{p,p}b$ do not equal zero. Such a vector b always exists, since $T_{p,p}$ is invertible. Then we choose $\lambda \in \mathbb{C}$ in such a way that each component of $\lambda b_0 + b$ is nonzero and that the last q components of Ta, where $a = (a_1, \dots, a_m)^t \in \mathbb{C}^m$ is defined by $a := (\lambda b_0 + b, 0, \dots, 0)^t$, are all nonzero. Note that the first $m - q$ components of Ta all equal 0.

Let us now define f by

$$f(j) = \sum_{k=0}^{m-1} a_{k+1} e^{2\pi i \frac{jk}{m}}.$$

An easy calculation shows that

$$\hat{f}(j) = \sum_{k=0}^{m-1} a_{k+1} \chi_{\{k\}}(j).$$

By choice of a, we have $m_G(A_f) = \frac{q}{m}$ and $\mu_G(B_f) = p$. □

We finish with a remark which states that we can easily extend Corollary 4.3 to general locally compact abelian groups.

Remark 4.6. Let G be a non-compact non-discrete locally compact abelian group such that G_0 is compact. Let H be a compact open subgroup of G. Suppose there exist $g \in L^1(H)$ and $r > 0$ such that $m_H(A_g)\mu_H(B_g) = r$. Then, for each $n \in \mathbb{N}$, we can construct some function $f \in L^1(G)$ such that

$$m_G(A_f)\mu_G(B_f) = nr.$$

This can be seen easily by choosing $x_i \in G$, $i = 1, \dots, n$ such that $x_iH \neq x_jH$ for all $i \neq j$ and defining $f \in L^1(G)$ by

$$f(x) = \sum_{i=1}^{n} a_i g(x_H) \chi_{x_iH}(x).$$

Here the decomposition $x = x_i x_H$ shall be unique for all $x \in \bigcup_{i=1}^n x_i H$ and the values $a_i \neq 0$, $i = 1, \ldots, n$ have to be chosen appropriately.

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